Limit linear systems and applications

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Abstract

A system of plane curves defined by prescribing n points of multiplicity e in general position is regular if $n \geq 4e^2$. The proof uses computation of limits of linear systems acquiring fixed divisors, an interesting problem in itself.

1 Introduction

Linear systems defined by multiple points in the plane are a classical object of study, still not well understood. Determining their regularity is one of the basic problems, equivalent to the solvability of bivariate Hermite interpolation problems and to the Riemann-Roch problem for rational surfaces. In spite of intense work devoted for decades to the question ([9], [29] and [39] are excellent overviews; even more recent results can be found in [6], [10], [22], [30], [32], [38], [47], [49], [51]) it is far from settled and the main conjectures remain open.

Given n points in the plane, and integers $e_1, \dots e_n$, the curves of degree d with multiplicity at least e_i at the i-th point form a linear system $\mathcal{L}$ of dimension at least

$$\frac{d(d+3)}{2} - \sum_{i=1}^{n} \frac{e_i(e_i+1)}{2}. \tag{1}$$

M. Nagata's famous conjecture of 1959 [40], motivated by his solution to Hilbert's 14th problem, states that a nonempty linear system $\mathcal{L}$ defined by points in general position must have $d > \left(\sum e_i\right)/\sqrt{n}$ if $n > 9$. $\mathcal{L}$ is called *regular* if it is empty or its dimension is given by (1). In 1961, B. Segre [48] conjectured that a linear system $\mathcal{L}$ defined by points in general position is either regular or has a multiple curve in its base locus. The Harbourne[26]–Hirschowitz[34] conjecture proposed in the 80's put further restrictions on the base curve; C. Ciliberto and R. Miranda proved [13] that it is equivalent to Segre's conjecture, and they both imply Nagata's conjecture.

In this work we focus on the equimultiple case $e_1 = \cdots = e_n = e$, assuming that the points are in general position. The Harbourne–Hirschowitz conjecture implies in this case that $\mathcal{L}$ is regular if $n \geq 9$ (note that the conjecture is known to be true for $n \leq 9$ [25], [27]). Our main result is the following.

Theorem 1.1. *Let e, n be positive integers with $n \geq 4e^2$. For general points $p_1, \dots, p_n \in \mathbb{P}^2$, and for every d, the linear system $\mathcal{L}$ of curves of degree d with multiplicity at least e at each p_i is regular.*

For comparison purposes, the only previous result which shows regularity for all d when $n \geq f(e)$ for some function f is due to J. Alexander and A. Hirschowitz [4], with $f \sim \exp(\exp(e))$.

Note also that regularity of $\mathcal{L}$ is known for all d and all $n \geq 9$ if $e \leq 42$ by recent work of M. Dumnicki [17], so $\mathcal{L}$ is regular whenever $e \leq \max\{42, \sqrt{n}/2\}$. Other known results for multiplicities small compared to the number of points support the weaker Nagata conjecture. Namely, in [32] it is proved that an equimultiple system $\mathcal{L}$ with $d \leq e\sqrt{n}$ is empty if $n \geq f(e)$, with $f(e) \sim e/2$ (L. Évain [20] proved a similar result with $f(e) \sim 2e^2$).

It is also worth mentioning that regularity is known to hold for small nonequal multiplicities in some cases as well; to begin with, the aforementioned Alexander–Hirschowitz result holds for nonequimultiple systems (and even in higher dimension), and M. Dumnicki–W. Jarnicki [18] have proved regularity for all d and all $n \geq 9$ if $e_i \leq 11 \forall i$. In a somewhat different spirit, S. Yang [51] proved that, given an upper bound $e_i \leq e \forall i$, there is a function $f(e) \sim e^2/\sqrt{6}$ such that, if $\mathcal{L}$ is regular for all d and all $n \in [9, f(e)]$, then $\mathcal{L}$ is regular for all d and all $n \geq 9$.

Let k be an algebraically closed field of arbitrary characteristic, and X a smooth projective variety over k. Given an invertible sheaf L and a zero-dimensional scheme $Z \subset X$, a natural generalization of the preceding considerations is to ask about the regularity of the system $|L - Z|$ of effective divisors in $|L| = \mathbb{P}(H^0(X, \mathcal{O}_X(L)))$ containing Z. Such a system is regular if the natural linear map (restriction)

$$\Gamma(X, \mathcal{O}_X(L)) \xrightarrow{\rho} \Gamma(Z, \mathcal{O}_Z(L))$$

has maximal rank, as $|L - Z| = \mathbb{P}(\ker \rho)$. It has revealed useful, when studying interpolation problems in general position, to consider families of schemes Z_t where the position of the points supporting Z_t varies with the parameter t. Then one obtains a family of maps ρ_t whose rank is lower semicontinuous in t, so it is enough to find one value of the parameter, say $t = 0$, where the rank is maximal, to conclude that it is so for general Z. Several specialization techniques employed both classically (see [41], [50], [40]) and recently (see [11], [24], [28], [33], [44], [45]) rely on the fact that, if enough of Z_0 lies on a divisor D of small degree, then all divisors in $|L - Z_0|$ must contain D, and subtracting D gives a linear system of the same dimension with smaller degree and smaller Z; then one hopes to show maximal rank inductively. A systematic use of specialization to divisors is sometimes called “the Horace method” after [33].

The drawback to this method is that if ρ_0 does not have maximal rank, it just gives a weak bound for the actual behaviour in general position. Alexander-Hirschowitz [2] (resumed and refined by the same authors in [3] and [4], by Mignon in [36], [37] and [38], by Chandler in [8] and by Évain in [19], [23]) and Ciliberto-Miranda in [11] (resumed and refined by the same authors in [12] and [14] and by Buckley-Zompatori in [6]) have shown a way around this obstacle. Denote by

$$\operatorname{edim} |L - Z| = \max\{\dim \Gamma(X, \mathcal{O}_X(L)) - \operatorname{length} Z - 1, -1\}$$

the expected dimension. The idea is to consider the limit of $|L - Z_t|$ when t tends to 0 (in the Grassmannian of $|L|$), and to construct a suitable “intermediate” interpolation problem

$$\Gamma(X, \mathcal{O}_X(L - mD)) \xrightarrow{\rho'_0} \Gamma(Z'_0, \mathcal{O}_{Z'_0}(L - mD)),$$

in the sense that

$$\lim_{t\mapsto 0} |L - Z_t| \subset mD + |(L - mD) - Z'_0| \subset |L - Z_0|$$

and $\operatorname{edim} |L - Z_t| = \operatorname{edim} |(L - mD) - Z'_0|$. Then it is enough that ρ'_0 have maximal rank (rather than ρ_0) to deduce that $\lim_{t\mapsto 0} |L - Z_t| = mD + |(L - mD) - Z'_0|$ and prove maximal rank for schemes in general position.

Here we explicit a method to systematically construct such intermediate problems, in the particular case that only one point of the support of Z_t varies with t, and study its range of applicability. The computation of limit linear systems is interesting on its own, in addition to our original motivation, for instance for the computation of limits in Hilbert schemes [23], and for adjacency of equisingularity types [1]. Hence we are interested in presenting the method in its natural general setting; also, even for the applications to the plane, we actually need to compute limit systems on other rational surfaces X.

Our approach is a generalization of the differential Horace method as presented by Évain in [19]. In [23] Évain gives a further generalization that allows several points to move, and even though his statements deal with vertical translations of monomial schemes only, our definition of the intermediate problem can be implicitly found in his proofs. However, the methods of [19] and [23] don't directly apply to general families (or even to families of monomial schemes moving non-vertically) because not every family allows an intermediate system of the same expected dimension as the original. It may even happen that the limit system is not determined by the condition of containing a subscheme. We have identified the obstructions to the existence of such intermediate systems for a general family of zero-dimensional schemes with one moving point, as elements in certain ideal quotients (proposition 2.4). Such obstructions did not appear in the method of [23], because they in fact vanish for vertical translations of a monomial scheme.

To effectively apply the method to a particular family of zeroschemes, some nontrivial algebraic computations are needed. Here we restrict to families of monomial zeroschemes (moving non-vertically) and their projections by blowing down. In a few cases we can then use the computations of [23]. We expect however to exploit the generality of the method in the future, as the knowledge of obstructions should help in the search of useful specializations.

1.1 Limit linear systems

Let C be a quasi-projective smooth curve over k and let Z be a subscheme of $X \times C$ which is flat and finite over C. The dimension of $|L - Z_t|$ is an upper semicontinuous function of $t \in C$ (with the Zariski topology of C). Thus there is an open set $U \subset C$ where $\dim |L - Z_t|$ is minimal and constant, say d. This gives a morphism to the Grassmannian of d-dimensional linear subspaces of $|L|$,

$$\begin{aligned} U &\to \mathbb{G}(|L|, d) \\ t &\mapsto |L - Z_t| \end{aligned}$$

which can be extended to the whole of C because the Grassmannian is projective. For $t_0 \notin U$ we denote $\lim_{t\mapsto t_0} |L - Z_t|$ the image of t_0 by the extension of the morphism above to C. Let $\mathcal{I}_t$ be the ideal sheaf of Z_t. Assume that for some

$t_0 \in C$ there exists a prime divisor $D \subset X$ such that $\rho^D_{t_0} : H^0(D, \mathcal{O}_D(L)) \to H^0(D \cap Z_{t_0}, \mathcal{O}_{D\cap Z_{t_0}}(L))$ is injective; then D is a fixed part of $|L - Z_{t_0}|$. The residual linear system after subtracting D, which has the same dimension as $|L - Z_{t_0}|$, is $|(L - D) - \tilde{Z}_{t_0}|$, with $\tilde{Z}_{t_0}$ defined by the residual exact sequence

$$0 \to \mathcal{I}_{\tilde{Z}_{t_0}} \to \mathcal{I}_{Z_{t_0}} \to \mathcal{I}_{Z_{t_0}} \otimes \mathcal{O}_D \to 0.$$

In order to use the special member Z_{t_0} to prove regularity of general $|L - Z_t|$, this residual should have the same expected dimension as the original systems. But if $\rho^D_{t_0}$ is not surjective, the expected dimension will jump (in the language of Horace methods, the specialization is not adjusted).

The jump in expected dimension comes from specializing "too much" of Z_{t_0} onto D. Now, the idea of [4] is, roughly speaking, to take $(t - t_0)^p = 0$ for some $p > 1$, so that $Z_t \cap D$ is big enough to have ρ^D_t injective, but not as much as $Z_{t_0} \cap D$, so that ρ^D_t can be adjusted. Generalizing this idea as in [23], if $\mathbf{p} = (p_1, \dots, p_m) \in \mathbb{Z}^m$ is a non-increasing sequence, one takes more and more special positions given by $(t - t_0)^{p_1} = 0$, $(t - t_0)^{p_2} = 0$, etc. Then one needs an adequate notion of residual to bound $\lim_{t\mapsto t_0} |L - Z_t|$ and show that these intuitions correspond rigorously to an actual phenomenon. In section 2 iterated trace $\mathrm{Tr}^i_{\mathbf{p}}(Z_t|D)$ and residual $\mathrm{Res}^i_{\mathbf{p}}(Z_t|D)$ ideals are defined, providing intermediate systems, and we show that under suitable hypotheses, including (but not restricted to) the specializations of monomial ideals of [2], [4], [36], and [23], they do have the same expected dimension as $|L - Z_t|$.

Suppose that $Z = (Z_{\text{fix}} \times C) \cup Y \subset X \times C$, where $Z_{\text{fix}} \subset X$ is a fixed zero-dimensional scheme and Y is irreducible, finite and flat over C. In other words, Z has a fixed and a moving part, and the moving part is supported at a single (possibly moving) point of X. Assume that Y_{t_0} is supported at a point on a prime dvisor D, and ρ^D_t is injective. Denoting by $\mathrm{SchTr}^i_{\mathbf{p}}(Z_t|D) \subset D$ and $\mathrm{SchRes}^i_{\mathbf{p}}(Z_t|D) \subset X$ the zero-dimensional subschemes defined by the trace and residual ideals, the main result on limit linear systems is the following:

Theorem 1.2. *Let $Z = (Z_{\text{fix}} \times C) \cup Y$ be as above, and take a sequence $\mathbf{p} = (p_1, \dots, p_m) \in \mathbb{Z}^m$ with $p_1 \geq p_2 \geq \cdots \geq p_m \geq 1$.*

1. *If for $1 \leq i \leq m$, the map*

   $$\rho^D_i : H^0(D, \mathcal{O}_D(L - (i-1)D)) \longrightarrow H^0(\mathrm{SchTr}^i_{\mathbf{p}}(Z_t|D), \mathcal{O}_{\mathrm{SchTr}^i_{\mathbf{p}}(Z_t|D)}(L))$$

   is injective, then $\lim_{t\mapsto t_0} |L - Z_t| \subset mD + |(L - mD) - \mathrm{SchRes}^m_{\mathbf{p}}(Z_t|D)|$.

2. *Given Z and $p_1, \dots, p_i$, $i < m$, there exists $1 \leq q_i$ such that, if for all $i = 1, \dots m - 1$, ρ^D_i is bijective, $p_{i+1} \leq q_i$ and the restriction map $H^0(X, \mathcal{O}_X(L - (i-1)D)) \to H^0(D, \mathcal{O}_D(L - (i-1)D))$ has maximal rank, then* $\mathrm{edim}\,|L - Z_t| = \mathrm{edim}\,|(L - mD) - \mathrm{Res}^m_{\mathbf{p}}(Z_t|D)|$.

The results of section 2 are in fact slightly more general, since we allow for singular and reducible fixed divisors $D = D_1 + \cdots + D_k$, and each component may appear with a different multiplicity in $|L - Z_{t_0}|$. The first claim of theorem 1.2 is a natural generalization of Évain [19], more or less implicit in [23, Theorem 1], but the second is to our knowledge entirely new, since the methods of [23] (which give $q_i = p_i - 1$ in the case of vertically translated monomial schemes) do not apply in the general setting.

Whereas the bijectivity hypotheses in the second claim of theorem 1.2 are adjustment requirements (depending on the global geometry of D), the hypotheses on $\mathbf{p}$ are of a new local kind: they force that the obstructions mentioned above and specified in proposition 2.4 actually vanish for the given specialization. The exact value of q_i may be found in the proof of corollary 2.5, but in the applications it will be advantadgeous to apply results of section 2.4, where sequences $\mathbf{p}$ are analyzed with respect to the valuative properties of Z and D_i.

The proof of theorem 1.1 is based on theorem 1.2. However, we don't actually compute the schemes $\mathrm{SchTr}^i_{\mathbf{p}}(Z_t|D) \subset D$ and $\mathrm{SchRes}^i_{\mathbf{p}}(Z_t|D) \subset X$; instead, we give bounds for them and use the second part of 1.2 to make sure that the expected dimension is preserved and so the regularity of the limit system proves regularity of the general ones.

Once theorem 1.1 is known, and using theorem 1.2, the following result of Évain [23] (see also Ciliberto-Miranda [15]) can be quickly proved:

Theorem 1.3. *Assume that the characteristic of the base field k is zero. Let $n = s^2$ be a square, and let $e \geq 1$ be an integer. The scheme formed by n distinct points in general position in $\mathbb{P}^2$ with multiplicity e has maximal rank in all degrees.*

An interesting feature of the proof of 1.3 is that it shows that the method can still be useful when obstructions do appear, to prove emptyness of a linear system. Also, it may be worth noting that it uses the same specialization Nagata used in [40] to prove that, for every square $n = s^2 > 9$ and every integer e there are no curves of degree se with multiplicity at least e at n general points.

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2 Algebraic approach to intermediate linear systems

Section 2.1 contains the algebraic local analysis of the behaviour of a linear system moving in a 1-dimensional family that acquires a base divisor (possibly reducible with multiple components) in a special position. In section 2.2 we determine under which conditions the intermediate linear system given by our method coincides with the expected limit linear system, and we prove theorem 1.2. To effectively apply the results of this section, some computations are needed which tend to be nontrivial. Section 2.3 shows one such computation, needed for the proof of theorem 1.3. Part of these computations can be arranged in a systematic way, and we obtain sufficient conditions under which

the intermediate linear system coincides with the expected limit system, for homogeneous or monomial families, in sections 2.4 and 2.5.

2.1 Higher order traces and residuals

Let R be an integral k-algebra, and consider $R_t = R \otimes k[[t]]$. Given $f_t \in R_t$, denote $f_0 \in R$ its image by the obvious morphism $t \mapsto 0$. Similarly, for an ideal I_t in R_t, $I_0 = (I_t + (t))/(t) \subset R_t/(t) \cong R$.

We define higher order traces and residuals of I_t on divisors $y = 0$, in the spirit of [23]. Loosely speaking, if the I_t define a family of schemes, we want to consider the trace on the divisor $y = 0$ of the special member given by $t^p = 0$, and compute the residual family (over $k[[t]]/(t^p)$) in a way that allows to consider the trace on another divisor $z = 0$ of every special residual given by $t^q = 0$, $q \leq p$, etc.

Given an ideal $I_t \subset R_t$, an element $y \in R$ and an integer $p \geq 1$, consider the following ideals:

$$\begin{aligned}
\mathrm{Tr}_p(I_t|y) =& \frac{\big((I_t + (y)) : t^{p-1}\big)_0}{(y)} \subset R/(y), \\
\mathbb{R}\mathrm{es}_p(I_t|y) =& (I_t + (t^p)) : y_1 \subset R_t, \\
\mathrm{Res}_p(I_t|y) =& ((I_t + (t^p)) : y)_0 \subset R.
\end{aligned}$$

Note that $\mathrm{Res}_p(I_t|y) = \big(\mathbb{R}\mathrm{es}_p(I_t|y)\big)_0$, and there are inclusions $\mathrm{Tr}_1(I_t|y) \subset \mathrm{Tr}_2(I_t|y) \subset \dots$, and $\mathrm{Res}_1(I_t|y) \supset \mathrm{Res}_2(I_t|y) \supset \dots \supset I_0$.

More generally, given sequences $\mathbf{p} = (p_1, \dots, p_m) \in \mathbb{Z}^m$ with $p_1 \geq p_2 \geq \dots \geq p_m \geq 1$, $\mathbf{y} = (y_1, \dots, y_m) \in R^m$, denote $\mathrm{Tr}^1_{\mathbf{p}}(I_t|\mathbf{y}) = \mathrm{Tr}_{p_1}(I_t|y_1)$, $\mathrm{Res}^1_{\mathbf{p}}(I_t|\mathbf{y}) = \mathrm{Res}_{p_1}(I_t|y_1)$ and $\mathbb{R}\mathrm{es}^1_{\mathbf{p}}(I_t|\mathbf{y}) = \mathbb{R}\mathrm{es}_{p_1}(I_t|y_1)$; then for every integer $1 < i \leq m$ define iteratively the following ideals:

$$\begin{aligned}
\mathrm{Tr}^i_{\mathbf{p}}(I_t|\mathbf{y}) =& \mathrm{Tr}_{p_i}(\mathbb{R}\mathrm{es}^{i-1}_{\mathbf{p}}(I_t|\mathbf{y})|y_i) \subset R/(y_i), \\
\mathbb{R}\mathrm{es}^i_{\mathbf{p}}(I_t|\mathbf{y}) =& \mathbb{R}\mathrm{es}_{p_i}(\mathbb{R}\mathrm{es}^{i-1}_{\mathbf{p}}(I_t|\mathbf{y})|y_i) \subset R_t, \\
\mathrm{Res}^i_{\mathbf{p}}(I_t|\mathbf{y}) =& \mathrm{Res}_{p_i}(\mathbb{R}\mathrm{es}^{i-1}_{\mathbf{p}}(I_t|\mathbf{y})|y_i) \subset R.
\end{aligned}$$

If $y_i = y \, \forall i$ we write $\mathrm{Tr}^i_{\mathbf{p}}(I_t|y) = \mathrm{Tr}^i_{\mathbf{p}}(I_t|\mathbf{y})$, $\mathbb{R}\mathrm{es}^i_{\mathbf{p}}(I_t|y) = \mathrm{Tr}^i_{\mathbf{p}}(I_t|\mathbf{y})$ and $\mathrm{Res}^i_{\mathbf{p}}(I_t|y) = \mathrm{Tr}^i_{\mathbf{p}}(I_t|\mathbf{y})$, and if $p_i = p \, \forall i$ we write $\mathrm{Tr}^i_p(I_t|\mathbf{y}) = \mathrm{Tr}^i_{\mathbf{p}}(I_t|\mathbf{y})$, etc. Note that

$$\mathbb{R}\mathrm{es}^i_{\mathbf{p}}(I_t|\mathbf{y}) = (I_t + (t^{p_1}, y_1 t^{p_2}, \ \dots \ , y_1 y_2 \cdots y_{i-1} t^{p_i})) : (y_1 y_2 \cdots y_i). \qquad (2)$$

Sometimes we shall also write $\mathbb{R}\mathrm{es}^0_{\mathbf{p}}(I_t|y) = I_t$ and $\mathrm{Res}^0_{\mathbf{p}}(I_t|y) = I_0$.

Proposition 2.1 below (or rather its immediate corollary 2.2) is a natural generalization of theorem 14 in [19] (proved in [23] for products of ideals in products of rings), which in turn refines proposition 8.1 of [4]. It justifies the definitions given so far, and it will imply the first part of theorem 1.2. The reader may notice that the method of proof is essentially the same used by Évain in [23], theorem 1, for the particular case that R is a power series ring, one of the variables is $y = y_i \, \forall i$, and I_t is a monomial ideal of R translated "vertically", i.e., by $y \mapsto y + t$. For this particular case, equivalent definitions to the ones above can be found in the proof of theorem 1 of [23] (in particular, $\mathbb{R}\mathrm{es}^i_{\mathbf{p}}(I_t|y)$ is called $J_{p_1:\dots:p_i:}$ there).

For a k-linear subspace $V \subset R$ and $y \in R$, let $\mathrm{Res}(V|y) = \{v \in R \mid vy \in V\}$.

Proposition 2.1. *Let $V \subset R$ be a k-linear subspace, and $I_t \subset R_t$ an ideal such that R_t/I_t is flat over $k[[t]]$. Let $m \geq 0$ and let $\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{Z}^m$ with $p_1 \geq p_2 \geq \cdots \geq p_m \geq 1$, and $\mathbf{y} = (y_1, \ldots, y_m) \in R^m$ be given sequences. Let $W = \{f \in R | \exists f_t \in V \otimes k[[t]] \cap I_t$ with $f_0 = f\}$. If for $1 \leq i \leq m$, the canonical map*

$$\frac{\operatorname{Res}(V|y_1 \cdots y_{i-1})}{\operatorname{Res}(V|y_1 \cdots y_{i-1}) \cap (y_i)} \longrightarrow \frac{R/(y_i)}{\operatorname{Tr}^i_{\mathbf{p}}(I_t|\mathbf{y})}$$

is injective, then $W \subset y_1 \ldots y_m \operatorname{Res}^m_{\mathbf{p}}(I_t|\mathbf{y})$.

Proof. Let $f_t \in V \otimes k[[t]] \cap I_t$. If

$$f_t \in (t^{p_1}, y_1 t^{p_2}, \ \ldots \ , y_1 y_2 \cdots y_{m-1} t^{p_m}, y_1 y_2 \cdots y_m), \tag{3}$$

i.e., if $f_t = g_t y_1 y_2 \cdots y_m + h_t$ for some $h_t \in (t^{p_1}, y_1 t^{p_2}, \ \ldots \ , y_1 y_2 \cdots y_{m-1} t^{p_m})$, then (2) implies $g_0 \in \operatorname{Res}(V|y_1 y_2 \cdots y_m) \cap \operatorname{Res}^m_{\mathbf{p}}(I_t|\mathbf{y})$, and therefore $f_0 \in y_1 \ldots y_m \operatorname{Res}^m_{\mathbf{p}}(I_t|\mathbf{y})$ as claimed. So it will be enough to prove (3).

We use induction on m. For $m = 0$, one has trivially $f_t \in (\prod_{i=1}^0 y_i) = R_t$. Assume now $m > 0$ and

$$f_t \in (t^{p_1}, y_1 t^{p_2}, \ \ldots \ , y_1 y_2 \cdots y_{m-2} t^{p_{m-1}}, y_1 y_2 \cdots y_{m-1}).$$

Denoting $p = p_m$, and taking into account that $p_1 \geq p_2 \geq \cdots \geq p_m \geq 1$, this implies $f_t \in (y_1 y_2 \cdots y_{m-1}, t^p)$, i.e., $f_t = g_t y_1 y_2 \cdots y_{m-1} + h_t t^p$, where we may further assume that $g_t = G_0 + G_1 t + \cdots + G_{p-1} t^{p-1}$, with $G_j \in R$, $j = 0, \ldots, p$. Denote $\bar{G}_j$ the class of G_j in $R/(y_m)$; we want to see that $\bar{G}_0 = \cdots = \bar{G}_{p-1} = 0$.

The inclusions $\operatorname{Tr}_0(\mathbb{R}\mathrm{es}^{m-1}_{\mathbf{p}}(I_t|\mathbf{y})|y_m) \subset \operatorname{Tr}_1(\mathbb{R}\mathrm{es}^{m-1}_{\mathbf{p}}(I_t|\mathbf{y})|y_m) \subset \ldots$ together with the hypothesis in the case $i = m$ tell us that, for every $j = 1, \ldots, p$, the map

$$\varphi_j : \frac{\operatorname{Res}(V|y_1 y_2 \cdots y_{m-1})}{\operatorname{Res}(V|y_1 y_2 \cdots y_{m-1}) \cap (y_m)} \longrightarrow \frac{R/(y_m)}{\operatorname{Tr}_j(\mathbb{R}\mathrm{es}^{m-1}_{\mathbf{p}}(I_t|\mathbf{y})|y_m)}$$

is injective. As we have $g_t \in \mathbb{R}\mathrm{es}^{m-1}_{\mathbf{p}}(I_t|\mathbf{y})$, it follows that

$$\bar{G}_0 \in \frac{(\mathbb{R}\mathrm{es}^{m-1}_{\mathbf{p}}(I_t|\mathbf{y}) + (y_m))_0}{(y_m)} = \operatorname{Tr}_0(\mathbb{R}\mathrm{es}^{m-1}_{\mathbf{p}}(I_t|\mathbf{y})|y_m),$$

i.e., $\varphi_0(\bar{G}_0) = 0$, and therefore $\bar{G}_0 = 0$. Now we argue by iteration: let $1 \leq j < p$, and assume we know $\bar{G}_0 = \cdots = \bar{G}_{j-1} = 0$. This means that $g_t \in (y_m, t^j)$, so $G_j + \cdots + G_{p-1} t^{p-1-j} \in (\mathbb{R}\mathrm{es}^{m-1}_{\mathbf{p}}(I_t|\mathbf{y}) + (y_m)) : t^j$, which implies $\bar{G}_j \in \operatorname{Tr}_{j+1}(\mathbb{R}\mathrm{es}^{m-1}_{\mathbf{p}}(I_t|\mathbf{y})|y_m)$, i.e., $\varphi_{j+1}(\bar{G}_j) = 0$, and therefore $\bar{G}_j = 0$. □

Corollary 2.2. *Let $V \subset R$ be a k-linear subspace, and $I_t \subset R_t$ an ideal such that R_t/I_t is flat over $k[[t]]$. Let $m \geq 0$ and let $\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{Z}^m$ with $p_1 \geq p_2 \geq \cdots \geq p_m \geq 1$, and $\mathbf{y} = (y_1, \ldots, y_m) \in R^m$ be given sequences. Suppose that the following two conditions are satisfied.*

1. *For $1 \leq i \leq m$, the canonical map*

$$\frac{\operatorname{Res}(V|y_1 \cdots y_{i-1})}{\operatorname{Res}(V|y_1 \cdots y_{i-1}) \cap (y_i)} \longrightarrow \frac{R/(y_i)}{\operatorname{Tr}^i_{\mathbf{p}}(I_t|\mathbf{y})}$$

   is injective.

2. The canonical map

$$\mathrm{Res}(V|y_1\cdots y_m) \longrightarrow \frac{R}{\mathrm{Res}^m_{\mathbf{p}}(I_t|\mathbf{y})}$$

is injective.

Then the canonical map $\varphi_t : V\otimes k[[t]] \longrightarrow R_t/I_t$ is injective.

Proof. Let $f_t \in \operatorname{Ker}\varphi_t = V\otimes k[[t]]\cap I_t$. If $f_0 = 0$, then we may replace f_t by f_t/t since R_t/I_t is a flat and hence torsion free $k[[t]]$-module. Thus we only have to prove $f_0 = 0$. But proposition 2.1 implies that $f_0 = gy_1\dots y_m$ for some $g\in \mathrm{Res}(V|y_1\cdots y_m)\cap \mathrm{Res}^m_{\mathbf{p}}(I_t|\mathbf{y})$, so the second hypothesis gives $g_0 = 0$ and therefore $f_0 = 0$. □

2.2 Preserving the number of conditions

We are mostly interested in flat families R_t/I_t of finite length (which define flat families of zeroschemes Z_t); so it will be useful to consider the quantities $\mathrm{tr}^i_{\mathbf{p}}(I_t|\mathbf{y}) = \dim_k(R/\mathrm{Tr}^i_{\mathbf{p}}(I_t|\mathbf{y}))$, $\mathbb{res}^i_{\mathbf{p}}(I_t|\mathbf{y}) = \dim_k(R_t/\mathbb{Res}^i_{\mathbf{p}}(I_t|\mathbf{y}))$, and $\mathrm{res}^i_{\mathbf{p}}(I_t|\mathbf{y}) = \dim_k(R/\mathrm{Res}^i_{\mathbf{p}}(I_t|\mathbf{y}))$. In particular, if R_t/I_t is flat over $k[[t]]$ then $\mathbb{res}^0_{\mathbf{p}}(I_t|y) = \infty$ and $\mathrm{res}^0_{\mathbf{p}}(I_t|y) = \dim_k R/I_0 = \dim_{k((t))}(R_t/I_t)\otimes k((t))$.

Our aim is to obtain a linear system $\mathcal{L}$ which contains the limit of a family of linear systems $|L-Z_t|$ and, if possible, coincides with it. In the best cases, this will serve to prove that general members of the family of linear systems are regular, i.e., of dimension equal to $\dim|L| - \operatorname{length} Z_t$, or -1 if this amount is negative.

In the approach of section 2.1, $\mathcal{L}$ consists of the elements of $|L|$ that contain (a) the divisors locally given by $y_1 = 0,\dots,y_m = 0$ (containing $y_i = 0$ accounts for $\mathrm{tr}^i_{\mathbf{p}}(I_t|\mathbf{y})$ linear conditions) and (b) the residual zeroscheme (which accounts for $\mathrm{res}^m_{\mathbf{p}}(I_t|\mathbf{y})$ linear conditions). So if $\dim|L| - \operatorname{length} Z_t \geq 0$, a requirement for the method to give the desired result is that

$$\mathrm{res}^m_{\mathbf{p}}(I_t|\mathbf{y}) + \sum_{i=1}^m \mathrm{tr}^i_{\mathbf{p}}(I_t|\mathbf{y}) = \mathrm{res}^0_{\mathbf{p}}(I_t|y), \tag{4}$$

and we now analyze when (4) is satisfied. Note that if it is not satisfied, the method can sometimes still be applied to prove that a linear system of interest is empty (see section 3.2).

Given a sequence $\mathbf{p} = (p_1,\dots,p_m)\in\mathbb{Z}^m$ and an integer $q\leq p_m$, define $\mathbf{p}-q = (p_1-q,\dots,p_m-q)$.

Lemma 2.3. *Let $I_t\subset R_t$ be an ideal such that R_t/I_t is flat over $k[[t]]$. Let $m\geq 0$ and let $\mathbf{p} = (p_1,\dots,p_m)\in\mathbb{Z}^m$ with $p_1\geq p_2\geq\cdots\geq p_m\geq 1$, and $\mathbf{y} = (y_1,\dots,y_m)\in R^m$ be given sequences. Then for every integer $q\leq p_m$,*

$$\mathbb{Res}^m_{\mathbf{p}}(I_t|\mathbf{y}) : t^q = \mathbb{Res}^m_{\mathbf{p}-q}(I_t|\mathbf{y}).$$

Proof. Using (2), it is easy to check that

$$\begin{aligned}\mathbb{Res}^m_{\mathbf{p}}(I_t|\mathbf{y}) : t^q &= (I_t + (t^{p_1}, y_1t^{p_2},\ \dots\ ,y_1y_2\cdots y_{m-1}t^{p_m})) : (y_1y_2\cdots y_m t^q) =\\ &= \left(I_t : t^q + (t^{p_1-q}, y_1t^{p_2-q},\ \dots\ ,y_1y_2\cdots y_{m-1}t^{p_m-q})\right) : (y_1y_2\cdots y_m).\end{aligned}$$

The claim follows noting that $I_t : t^q = I_t$ (by flatness) and using (2) again. □

It is well known that, for every ideal $I \subset R$, where R is a domain, and every $f \in R$, there is an exact sequence

$$0 \longrightarrow \frac{R}{I : f} \longrightarrow \frac{R}{I} \longrightarrow \frac{R}{I + (f)} \longrightarrow 0, \tag{5}$$

which we call *the residual exact sequence of I with respect to f*. Given a sequence $\mathbf{p} = (p_1, \dots, p_m) \in \mathbb{Z}^m$ and two integers j and q, with $1 \le j < m$ and $q \le p_j$, let us denote $\mathbf{p}(q, j) = (p_1, p_2, \dots, p_{j-1}, q)$.

Proposition 2.4. *Let $I_t \subset R_t$ be an ideal such that R_t/I_t is flat over $k[[t]]$ and $\dim_k R/I_0 < \infty$. Let $m \ge 0$ and let $\mathbf{p} = (p_1, \dots, p_m) \in \mathbb{Z}^m$ with $p_1 \ge p_2 \ge \dots \ge p_m \ge 1$, and $\mathbf{y} = (y_1, \dots, y_m) \in R^m$ be given sequences. Then*

1. *for every $i = 1, \dots, m$,*

$$\operatorname{res}^i_{\mathbf{p}}(I_t|\mathbf{y}) + \sum_{j=1}^{i} \operatorname{tr}^j_{\mathbf{p}}(I_t|\mathbf{y}) = \operatorname{res}^0_{\mathbf{p}}(I_t|\mathbf{y}) -$$

$$- \sum_{j=1}^{i} \dim \frac{\mathbb{R}\mathrm{es}^{j-1}_{\mathbf{p}-1}(I_t|\mathbf{y}) + (t^{p_j-1}, y_j)}{\mathbb{R}\mathrm{es}^{j-1}_{\mathbf{p}}(I_t|\mathbf{y}) + (t^{p_j-1}, y_j)},$$

2. *for every $j = 1, \dots, i$,*

$$\dim \frac{\mathbb{R}\mathrm{es}^{j-1}_{\mathbf{p}-1}(I_t|\mathbf{y}) + (t^{p_j-1}, y_j)}{\mathbb{R}\mathrm{es}^{j-1}_{\mathbf{p}}(I_t|\mathbf{y}) + (t^{p_j-1}, y_j)} = \sum_{q=1}^{p_j-1} \left(\operatorname{tr}^j_{\mathbf{p}(q,j)}(I_t|\mathbf{y}) - \operatorname{tr}^j_{(\mathbf{p}-1)(q,j)}(I_t|\mathbf{y}) \right).$$

The first claim of proposition 2.4 gives the amount by which the higher traces and residuals of I with respect to $\mathbf{y}$ fail to preserve the number of conditions imposed to the linear system. The second shows that this amount can be exactly computed whenever we can compute the colengths of the traces (even if the residuals are unknown).

Proof. Applying the residual exact sequence of $\mathbb{R}\mathrm{es}^{i-1}_{\mathbf{p}}(I_t|\mathbf{y}) + (t^{p_i}) \subset R_t$ with respect to $y_i \in R_t$ gives

$$\mathbb{r}\mathrm{es}^i_{\mathbf{p}}(I_t|\mathbf{y}) = \dim \frac{R_t}{\mathbb{R}\mathrm{es}^{i-1}_{\mathbf{p}}(I_t|\mathbf{y}) + (t^{p_i})} - \dim \frac{R_t}{\mathbb{R}\mathrm{es}^{i-1}_{\mathbf{p}}(I_t|\mathbf{y}) + (t^{p_i}, y_i)}.$$

The two terms on the right can be evaluated by means of residual exact sequences. Indeed, the residual exact sequence of $\mathbb{R}\mathrm{es}^{i-1}_{\mathbf{p}}(I_t|\mathbf{y}) + (t^{p_i})$ with respect to t^{p_i-1}, together with lemma 2.3, gives

$$\dim \frac{R_t}{\mathbb{R}\mathrm{es}^{i-1}_{\mathbf{p}}(I_t|\mathbf{y}) + (t^{p_i})} = \dim \frac{R_t}{\mathbb{R}\mathrm{es}^{i-1}_{\mathbf{p}}(I_t|\mathbf{y}) + (t^{p_i-1})} + \operatorname{res}^{i-1}_{\mathbf{p}-p_i+1}(I_t|\mathbf{y}),$$

and recursively applying this last equality,

$$\dim \frac{R_t}{\mathbb{R}\mathrm{es}^{i-1}_{\mathbf{p}}(I_t|\mathbf{y}) + (t^{p_i})} = \sum_{q=1}^{p_i} \operatorname{res}^{i-1}_{\mathbf{p}-q+1}(I_t|\mathbf{y}).$$

On the other hand, applying the residual exact sequence of $\mathbb{R}\mathrm{es}_{\mathbf{p}}^{i-1}(I_t|\mathbf{y}) + (t^{p_i}, y_i)$ with respect to t^{p_i-1} we get

$$\dim \frac{R_t}{\mathbb{R}\mathrm{es}_{\mathbf{p}}^{i-1}(I_t|\mathbf{y}) + (t^{p_i}, y_i)} = \dim \frac{R_t}{\mathbb{R}\mathrm{es}_{\mathbf{p}}^{i-1}(I_t|\mathbf{y}) + (t^{p_i-1}, y_i)} + \mathrm{tr}_{\mathbf{p}}^i(I_t|\mathbf{y}), \quad (6)$$

and so

$$\mathbb{r}\mathrm{es}_{\mathbf{p}}^i(I_t|\mathbf{y}) = \sum_{q=1}^{p_i} \mathrm{res}_{\mathbf{p}-q+1}^{i-1}(I_t|\mathbf{y}) - \mathrm{tr}_{\mathbf{p}}^i(I_t|\mathbf{y}) - \dim \frac{R_t}{\mathbb{R}\mathrm{es}_{\mathbf{p}}^{i-1}(I_t|\mathbf{y}) + (t^{p_i-1}, y_i)}.$$

Now apply the residual exact sequence of $\mathbb{R}\mathrm{es}_{\mathbf{p}}^i(I_t|\mathbf{y})$ with respect to t and lemma 2.3 to obtain that $\mathrm{res}_{\mathbf{p}}^i(I_t|\mathbf{y}) = \mathbb{r}\mathrm{es}_{\mathbf{p}}^i(I_t|\mathbf{y}) - \mathbb{r}\mathrm{es}_{\mathbf{p}-1}^i(I_t|\mathbf{y})$.

Putting together everything we have so far, it follows that

$$\begin{aligned}\mathrm{res}_{\mathbf{p}}^i(I_t|\mathbf{y}) &= \mathrm{res}_{\mathbf{p}}^{i-1}(I_t|\mathbf{y}) - \mathrm{tr}_{\mathbf{p}}^i(I_t|\mathbf{y}) - \\ &- \left(\dim \frac{R_t}{\mathbb{R}\mathrm{es}_{\mathbf{p}}^{i-1}(I_t|\mathbf{y}) + (t^{p_i-1}, y_i)} - \dim \frac{R_t}{\mathbb{R}\mathrm{es}_{\mathbf{p}-1}^{i-1}(I_t|\mathbf{y}) + (t^{p_i-1}, y_i)}\right),\end{aligned}$$

which recursively applied yields the first claim. The second follows by applying (6) recursively. □

One implication of proposition 2.4 is that the number of conditions is preserved whenever the integers in the sequence $\mathbf{p}$ decrease "fast enough", which will give the second part of theorem 1.2. We prove this next:

Corollary 2.5. *Assume that*

$$\mathrm{res}_{\mathbf{p}}^i(I_t|\mathbf{y}) + \sum_{j=1}^{i} \mathrm{tr}_{\mathbf{p}}^j(I_t|\mathbf{y}) = \mathrm{res}_{\mathbf{p}}^0(I_t|\mathbf{y}).$$

Then there exists an integer q_i, $p_i \geq q_i \geq 0$ such that

$$\mathrm{res}_{\mathbf{p}}^{i+1}(I_t|\mathbf{y}) + \sum_{j=1}^{i+1} \mathrm{tr}_{\mathbf{p}}^j(I_t|\mathbf{y}) = \mathrm{res}_{\mathbf{p}}^0(I_t|\mathbf{y}) \text{ if and only if } p_{i+1} \leq q_i.$$

Proof. The hypothesis, together with proposition 2.4, tell us that

$$\begin{aligned}&\mathrm{res}_{\mathbf{p}}^{i+1}(I_t|\mathbf{y}) + \sum_{j=1}^{i+1} \mathrm{tr}_{\mathbf{p}}^j(I_t|\mathbf{y}) = \\ &\qquad = \mathrm{res}_{\mathbf{p}}^0(I_t|\mathbf{y}) - \dim \frac{\mathbb{R}\mathrm{es}_{\mathbf{p}-1}^i(I_t|\mathbf{y}) + (t^{p_{i+1}-1}, y_{i+1})}{\mathbb{R}\mathrm{es}_{\mathbf{p}}^i(I_t|\mathbf{y}) + (t^{p_{i+1}-1}, y_{i+1})} = \\ &\qquad\qquad = \mathrm{res}_{\mathbf{p}}^0(I_t|\mathbf{y}) - \sum_{q=1}^{p_i-1} \left(\mathrm{tr}_{\mathbf{p}(q)}^i(I_t|\mathbf{y}) - \mathrm{tr}_{(\mathbf{p}-1)(q)}^i(I_t|\mathbf{y})\right).\end{aligned}$$

Now, for all q it is easy to see that $\mathrm{Tr}_{\mathbf{p}(q)}^i(I_t|\mathbf{y}) \subseteq \mathrm{Tr}_{(\mathbf{p}-1)(q)}^i(I_t|\mathbf{y})$, so clearly $q_i = \min\left\{q \,\middle|\, \mathrm{Tr}_{\mathbf{p}(q)}^i(I_t|\mathbf{y}) \neq \mathrm{Tr}_{(\mathbf{p}-1)(q)}^i(I_t|\mathbf{y})\right\} - 1$ satisfies the claim. □

Example 2.6. It follows from corollary 2.5 that non iterated traces and residuals (i.e., when $m = 1$, which is the case used in [4]) always preserve the number of conditions. On the other hand, it follows from [19], [23] that for monomial schemes approaching a unique divisor $y = 0$ vertically, $q_i = p_i - 1$. The simplest examples in which $p_{i+1} < p_i$ for all i but the number of conditions is not preserved involve monomial schemes moving non-vertically. Let $f = x + y + t \in R_t = k[[x, y]][[t]]$, $I = (f, x^2)^4$, $\mathbf{y} = (y, y, x)$, and $\mathbf{p} = (8, 7, p_3)$. It is not hard to compute $q_2 = 5$; therefore the number of conditions is not preserved if $\mathbf{p} = (8, 7, 6)$.

Proof of theorem 1.2. Let R be the local ring of X at the support point of Y_{t_0} (or its completion), y_i a local equation of D, V the image of the natural map $\mathcal{O}_X(L) \to R$ and $k[[t]]$ the completion of the local ring of C at t_0. Then proposition 2.1 gives the first part of the statement.

The second part of the statement follows from 2.5. Indeed, assume ρ_i^D is bijective for all i. If $H^0(X, \mathcal{O}_X(L-(i-1)D)) \to H^0(D, \mathcal{O}_D(L-(i-1)D))$ has maximal rank for all i and is injective for some i, then $\operatorname{edim}|L - Z_t| = \operatorname{edim}|(L - mD) - \operatorname{Res}^m_{\mathbf{p}}(I_t|\mathbf{y})| = \dim |L - mD| = 0$, and if $H^0(X, \mathcal{O}_X(L-(i-1)D)) \to H^0(D, \mathcal{O}_D(L-(i-1)D))$ is surjective for all i, then $\dim|L - iD| = \dim|L-(i-1)D| - \operatorname{length}\operatorname{SchTr}^i_{\mathbf{p}}(Z_t|D)$. If $p_{i+1} \le q_i$ with q_i as in 2.5, then

$$\operatorname{SchRes}^i_{\mathbf{p}}(Z_t|D) + \sum_{i=1}^{m} \operatorname{tr}^i_{\mathbf{p}}(Z_t|D) = \operatorname{res}^0_{\mathbf{p}}(Z_t|D)$$

so the claim follows. ☐

2.3 A computation for squares

Note that even if the number of conditions is not preserved, the method can still be useful to show that a linear system is empty. We illustrate this with a computation in $R_t = k[[x, y]][[t]]$ (proposition 2.10) needed for the proof of theorem 1.3. To simplify, consider the leading, or dominant, terms of series in the local ring $k[[x, y]][[t]]$, with respect to some regular system of parameters of the form $\{x, f, t\}$. Given $g = \sum a_{ijk} x^i f^j t^k \in k[[x, f]][[t]] = k[[x, y]][[t]]$, we set $\operatorname{ord}_t(g) = \min\{k | \exists i, j;\ a_{ijk} \neq 0\}$ and define the dominant part of g as $g^* = \sum a_{i,j,\operatorname{ord}_t(g)} x^i f^j t^{\operatorname{ord}_t(g)}$. An ideal $I_t \subset k[[x, y, t]]$ determines its ideal of dominant terms $I_t^* = (g^*)_{g \in I_t} \subset k[[x, y, t]]$. Observe that $(I_t^*)_0 = I_0$.

Lemma 2.7. *Let $f = y - t \in k[[x, y]][[t]]$, $I_t \subset k[[x, y, t]]$ and $p \in \mathbb{Z}$, $p \ge 1$. Assume that $I_t^* \subset (x, f)^m + (t^p)$. Then $((I_t + (t^p)) : y)^* \subset (x, f)^{m-1} + (t^p)$*

Proof. Let $g = \sum a_{ijk} x^i f^j t^k \in k[[x, f]][[t]]$ be such that $yg \in I_t + (t^p)$. We want to see that $g^* \in (x, f)^{m-1} + (t^p)$. But $yg = (f + t)g = \sum (a_{ijk} + a_{i,j,k-1}) x^i f^j t^k$ (where we set $a_{i,j,-1} = 0$ for all i, j) so $(yg)^* = f(g^*)$. On the other hand, $(yg)^* \in (I_t + (t^p))^* = I_t^* + (t^p) \subset (x, f)^m + (t^p)$. Both things together tell us that $g^* \in ((x, f)^m + (t^p)) : f = (x, f)^{m-1} + (t^p)$, as claimed. ☐

Proposition 2.8. *Let e, p, be positive integers with $e + 1 \ge p$, and let $I_t = (x, y - t) \subset k[[x, y, t]]$. Then*

1. *For every positive integer i, $\operatorname{tr}^i_p(I^e_t|y) \ge e + 2 - p - i$, and*

2. Assume k has characteristic zero. Then $\operatorname{Res}_p^e(I_t^e|y) \subset \mathfrak{m}^{\lfloor \frac{p}{2} \rfloor}$.

Proof. Using the previous lemma $i-1$ times, it follows that $\mathbb{Res}_{p+1}^{i-1}(I_t^e|y)+(t) \subset I_t^{e-i+1}+(t)$. On the other hand, it is easy to see (and is proved as part of proposition 8.1 in [4]) that $\operatorname{tr}_p(I_t^{e-i+1}|y) = e+2-p-i$, whence the first claim.

Because of (2), what remains to prove is

$$((I_t^e + (t^p)) : y^e)_0 \subset (x,y)^{\lfloor \frac{p}{2} \rfloor}.$$

Define again $f = y - t$ and consider the automorphism φ of R_t defined by $\varphi(x) = x$, $\varphi(y) = f$, $\varphi(t) = t$. It is a $k[[t]]$-automorphism (it leaves $k[[t]]$ fixed) and for every $g = \sum a_{ijk}x^iy^jt^k \in k[[x,y,t]]$, $\varphi(g)_0 = \sum a_{ij0}x^iy^j = g_0$. So the claim is equivalent to

$$\left(\left(\varphi^{-1}(I_t^e) + (t^p)\right) : \varphi^{-1}(y)^e\right)_0 = (((x,y)^e + (t^p)) : (y+t)^e)_0 \subset (x,y)^{\lfloor \frac{p}{2} \rfloor},$$

i.e., if $g(y+t)^e \in (x,y)^e + (t^p)$ then we need to prove $g_0 \in (x,y))^{\lfloor \frac{p}{2} \rfloor}$. But now the ideal $\varphi^{-1}(I_t^e)+(t^p) = (x,f)^e+(t^p)$ is monomial, and so $h = \sum b_{ijk}x^iy^jt^k \in (x,y)^e + (t^p)$ if and only if $b_{ijk} = 0$ for all $k < p$ and $i+j < e$.

Let now $g = \sum a_{ijk}x^iy^jt^k$ and assume that $h = g(y+t)^e = \sum b_{ijk}x^iy^jt^k \in (x,y)^e + (t^p)$. By definition,

$$h = \sum a_{ijk} \sum_{\ell=0}^{e} \binom{e}{\ell} x^i e^{\ell+j} t^{k+m-\ell},$$

so

$$b_{ijk} = \sum_{\ell=\ell_0}^{\ell_1} \binom{e}{\ell} a_{i,j-\ell,k+\ell-e},$$

where $\ell_0 = \max\{e-k, 0\}$ and $\ell_1 = \max\{j, e\}$. The condition $h \in I_t^e$ thus translates into the linear equations

$$\sum_{\ell=e-k}^{j} \binom{e}{\ell} a_{i,j-\ell,k+\ell-e}, \qquad 0 \le i,j,k;\; k < p;\; i+j \le e-1.$$

Some among these equations involve the same set of coefficients; namely, for each fixed i and $r = j+k-e$ satisfying $0 \le i \le e-1$ and $0 \le r < p-i-1$ we have obtained a system of linear equations

$$\begin{pmatrix} \binom{e}{e-p+1} & \binom{e}{e-p+2} & \cdots & \binom{e}{r} \\ \binom{e}{e-p+2} & \binom{e}{e-p+3} & \cdots & \binom{e}{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{e}{e-1-i-r} & \binom{e}{e-i-r} & \cdots & \binom{e}{e-1-i} \end{pmatrix} \begin{pmatrix} a_{i,r,0} \\ a_{i,r-1,1} \\ \vdots \\ a_{i,0,r} \end{pmatrix} = 0, \tag{7}$$

which, if $e-i-1-r-(e-p+1) \ge r$, admits only the trivial solution because the matrix on the left has nonzero determinant (see lemma 2.9 below). In particular, if $h \in I_t^e$ then for every (i,r) with $i+r < \lfloor \frac{p}{2} \rfloor$ (which trivially implies $e-i-1-r-(e-p+1) \ge r$) we obtain $a_{i,r,0} = 0$, which means $g_0 \in (x,y)^{\lfloor \frac{e}{2} \rfloor}$, and the second claim follows. $\square$

Lemma 2.9. *For every triple of integers $e \geq r \geq n \geq 1$, the following symmetric $(n+1) \times (n+1)$ matrix is invertible.*

$$H_{r,n}(e) = \begin{pmatrix} \binom{e}{r-n} & \binom{e}{r-n+1} & \cdots & \binom{e}{r} \\ \binom{e}{r-n+1} & \binom{e}{r-n+2} & \cdots & \binom{e}{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{e}{r} & \binom{e}{r+1} & \cdots & \binom{e}{r+n} \end{pmatrix}$$

Similar matrices are known in the litterature, e.g., in [5, p. 94], [42], [35].

Proof. For r, n fixed, $\det H_{r,n}(e) \in \mathbb{Q}[e]$ is a polynomial of degree (at most) $r(n+1)$, since the entry in the (i,j) position, $0 \leq i, j \leq n$, is

$$\binom{e}{r-n+i+j} = \frac{\prod_{a=0}^{r-n+i+j-1}(e-a)}{(r-n+i+j)!},$$

a polynomial of degree $r-n+i+j$. Moreover, every element of the ith row is divisible by $\binom{e}{r-n+i}$, and therefore $\det H_{r,n}(e)$ is divisible by $P(e) = \prod_{i=0}^{n} \binom{e}{r-n+i}$. On the other hand, by using the identities

$$\binom{e}{a} + \binom{e}{a+1} = \binom{e+1}{a+1},$$

a few elementary operations on rows show that

$$\det H_{r,n}(e) = \det \begin{pmatrix} \binom{e}{r-n} & \binom{e}{r-n+1} & \cdots & \binom{e}{r} \\ \binom{e+1}{r-n+1} & \binom{e+1}{r-n+2} & \cdots & \binom{e+1}{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{e+n}{r} & \binom{e+n}{r+1} & \cdots & \binom{e+n}{r+n} \end{pmatrix}.$$

Taking again common divisors to elements in each row one gets that $\det H_{r,n}(e)$ is divisible by $Q(e) = \prod_{i=0}^{n} \binom{e+i}{r-n+i}$. Therefore $\det H_{r,n}(e)$ is divisible by the l. c. m. of P and Q, which has degree $r(n+1)$ and all its roots in the set $\{-n, -n+1, \ldots, r-1\}$. It follows that all roots of $\det H_{r,n}(e)$ are strictly less than r, and thus for $e \geq r$, this determinant does not vanish. $\square$

Proposition 2.10. *Assume that the characteristic of the base field k is zero. Let $n = s^2$ be an odd square, and let $e > s/2$ be an integer. Let $a = se + (s-5)/2$. Given a set of n distinct points in general position in $\mathbb{P}^2$, there are no curves of degree a with multiplicity at least e at each point.*

Proof. Consider an irreducible smooth curve C of degree s (and genus $g = (s-1)(s-2)/2$), and let $p_1, \ldots, p_{s^2-1}$ be general points of C, whereas p_{s^2} is a general point of $\mathbb{P}^2$. Denote by Z the union of these points taken with multiplicity e. The restriction of $\mathcal{I}_Z(a)$ to C is an invertible sheaf of degree $d = as - (s^2-1)m = m - s(s-5)/2$ which, by the genericity of the choice of the $s^2 - 1 > g$ points, is general among those of its degree.

If $d < g$ then this invertible sheaf has no nonzero global sections, i.e., the curve C is a fixed part of the linear system $H^0(\mathcal{I}_Z(a))$. The residual linear system is formed by curves of degree $a - s = s(e-1) + (s-5)/2$ which contain the scheme Z' consisting of the points have $p_1, \ldots, p_{s^2-1}$ with multiplicity $e-1$

and p_{s^2} with multiplicity e. But then the restriction of $\mathcal{I}_{Z'}(a-s)$ to C is an invertible sheaf of degree $d' = d-1 < g$, and is still general among those of its degree, so C is again a fixed part of the linear system. Iterating this process, we see that C is contained exactly e times in the curves of the linear system $H^0(\mathcal{I}_Z(a))$, and the residual linear system consists of curves of degree $a-sm=(s-5)/2$ with a point of multiplicity $e > s/2$ so it is empty as claimed.

So assume $d \geq g$ and let $p = e+g-d = s+2$ (so we trivially have $e+1 \geq p \geq 1$). Now let p_{s^2} tend to C transversely. In other words, choose a (general) point $q \in C$, let x, y be local parameters at q such that $y=0$ is a local equation for C, and let $q_t = (0,-t)$. We want to see that the limit of the linear systems formed by curves of degree a with multiplicity e at $p_1, \ldots, p_{s^2-1}$ and at q_t when $t \mapsto 0$ is empty. Applying the first claim of 2.8 and 2.1, it follows that the limit linear system consists of C counted e times plus a moving part, consisting of curves of degree $a-es=(s-5)/2$ going through the zeroscheme defined by $\operatorname{Res}^e_{p+1}(I^e_t|y)$. But by the second claim of 2.8 this zeroscheme contains the point q counted $p/2 > s/2$ times, and we are done. $\square$

2.4 Homogeneous ideals in power series rings

Throughout this section we assume that $R = k[[x_1, \ldots, x_r]]$ is a power series ring, so both R and R_t are regular local rings, whose maximal ideals are $\mathfrak{m} = (x_1, \ldots, x_r)$ and $\mathfrak{m}_t = (x_1, \ldots, x_r, t)$ respectively, and come endowed with a natural $\mathfrak{m}$-adic valuation v.

Let $I_t \subset R_t$ be a homogeneous ideal, and $\mathbf{y} = (y_1, \ldots, y_m) \in R_t^m$ a sequence of homogeneous polynomials. All higher traces and residuals defined above are then homogeneous. Moreover, it is easy to see that if I and J are homogeneous ideals and f is a homogeneous polynomial, then for every integer e

$$I + \mathfrak{m}^e = J + \mathfrak{m}^e \ \Rightarrow I : f + \mathfrak{m}^{e-v(f)} = J : f + \mathfrak{m}^{e-v(f)}. \tag{8}$$

Thus higher traces and residuals do not differ from ordinary traces and residuals (up to a finite order that can be computed). So we will substitute one for the other to avoid too cumbersome computations (with due cautions, essentially contained in the following proposition).

Proposition 2.11. *Let $I_t \subset R_t = k[[x_1, \ldots, x_r, t]]$ be a homogeneous ideal. Let $m \geq 0$ and let $\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{Z}^m$ and $\mathbf{y} = (y_1, \ldots, y_m) \in R^m$ be given sequences, with y_i homogeneous for all i, such that $p_i - p_{i+1} \geq v(y_i)$ for $i = 1, \ldots, m-1$ and $p_m \geq v(y_m)$. For each $j = 1, \ldots, m$, define $\mathbf{y}_j = (y_j, \ldots, y_m)$ and $\mathbf{p}_j = (p_j, \ldots, p_m)$; for $1 \leq j \leq i \leq m$, let $V^i_j = \sum_{\ell=j}^{i} v(y_\ell)$. Then,*

1. *for every $1 \leq j \leq i \leq m$,*
$$\mathbb{R}\mathrm{es}^i_{\mathbf{p}}(I_t|\mathbf{y}) + \mathfrak{m}_t^{p_j - V^i_j} = \mathbb{R}\mathrm{es}^{i-j}_{\mathbf{p}_j}(I_t : (y_1 \ldots y_j)|\mathbf{y}_j) + \mathfrak{m}_t^{p_j - V^i_j},$$
2. *for every $1 \leq j < i \leq m$,*
$$\mathrm{Tr}^i_{\mathbf{p}}(I_t|\mathbf{y}) + \mathfrak{m}^{p_j - p_i + 1 - V^{i-1}_j} = \mathrm{Tr}^{i-j}_{\mathbf{p}_j}(I_t : (y_1 \ldots y_j)|\mathbf{y}_j) + \mathfrak{m}^{p_j - p_i + 1 - V^{i-1}_j}.$$

Proof. Observe that the first claim, together with (8), implies the second. Again, using (8) $i-j$ times we see that the first claim will follow from

$$\mathbb{R}\mathrm{es}^j_{\mathbf{p}}(I_t|\mathbf{y}) + \mathfrak{m}_t^{p_j - v(y_j)} = I_t : (y_1 \ldots y_j) + \mathfrak{m}_t^{p_j - v(y_j)},$$

which we prove by induction on j. For $j = 0$, there is nothing to prove. For $j > 0$, since $p_j \leq p_{j-1} - v(y_{j-1})$ the induction hypothesis gives

$$\mathbb{R}\mathrm{es}^{j-1}_{\mathbf{p}}(I_t|\mathbf{y}) + (t^{p_j}) + \mathfrak{m}_t^{p_j} = \mathbb{R}\mathrm{es}^{j-1}_{\mathbf{p}}(I_t|\mathbf{y}) + \mathfrak{m}_t^{p_j} = I_t : (y_1 \dots y_{j-1}) + \mathfrak{m}_t^{p_j},$$

which using (8) and the definition of $\mathbb{R}\mathrm{es}^{j}_{\mathbf{p}}(I_t|\mathbf{y})$ finishes the proof. □

The following technical lemma, which is a rather straightforward application of proposition 2.11, is the key to show that the number of conditions is preserved in all the applications contained in this paper. Given an ideal $I \subset R_t = k[[x_1, \dots, x_r, t]]$ and sequences $\mathbf{p} = (p_1, \dots, p_m) \in \mathbb{Z}^m$ and $\mathbf{y} = (y_1, \dots, y_m) \in R^m$, denote

$$\widetilde{\mathrm{tr}}^i_{\mathbf{p}}(I|\mathbf{y}) = \max_{1 \leq q \leq p_i} \{\mathrm{tr}^i_{\mathbf{p}(q,i)}(I|\mathbf{y}) + q - p_i\}.$$

Lemma 2.12. *Let $I_t \subset R_t = k[[x_1, \dots, x_r, t]]$ be a homogeneous ideal such that R_t/I_t is flat over $k[[t]]$. Let $m \geq 0$ and let $\mathbf{p} = (p_1, \dots, p_m) \in \mathbb{Z}^m$ and $\mathbf{y} = (y_1, \dots, y_m) \in R^m$ be given sequences, with y_i homogeneous for all i, such that $p_i - p_{i+1} \geq v(y_i)$ for $i = 1, \dots, m-1$ and $p_m \geq v(y_m)$. For each $j = 1, \dots, m$, define $\mathbf{y}_j = (y_j, \dots, y_m)$ and $\mathbf{p}_j = (p_j, \dots, p_m)$; for $1 \leq j \leq i \leq m$, let $V^i_j = \sum_{\ell=j}^{i} v(y_\ell)$. Assume that for every $i > 1$ there is $j < i$ with*

1. $p_j - p_i \geq V^{i-1}_j + \widetilde{\mathrm{tr}}^{i-j}_{\mathbf{p}_i(p_i-1)}(I_t : (y_1 \dots y_j)|\mathbf{y}_j) - 2$,
2. $\mathrm{res}^{i-j}_{\mathbf{p}_j}(I_t : (y_1 \dots y_j)|\mathbf{y}_j) + \sum_{\ell=i+1}^{j} \mathrm{tr}^{\ell}_{\mathbf{p}_j}(I_t : (y_1 \dots y_j)|\mathbf{y}_j) =$
   $= \mathrm{res}^0_{\mathbf{p}_j}(I_t : (y_1 \dots y_j)|\mathbf{y}_j)$.

Then

$$\mathrm{res}^m_{\mathbf{p}}(I_t|\mathbf{y}) + \sum_{i=1}^{m} \mathrm{tr}^i_{\mathbf{p}}(I_t|\mathbf{y}) = \mathrm{res}^0_{\mathbf{p}}(I_t|\mathbf{y}).$$

Proof. Due to proposition 2.4, what is needed to show is that for each $i = 1, \dots, m$ and $q = 1, \dots, p_i - 1$, $\mathrm{tr}^i_{\mathbf{p}(q)}(I_t|\mathbf{y}) = \mathrm{tr}^i_{(\mathbf{p}-1)(q)}(I_t|\mathbf{y})$. By hypothesis there is $j < i$ with $p_j - q - \sum_{\ell=j}^{i-1} v(y_\ell) + 1 \geq \mathrm{tr}^{i-j}_{\mathbf{p}_j(q)}(I_t : (y_1 \dots y_j)|\mathbf{y}_j)$ for all $1 \leq q \leq p_i - 1$, and therefore $\mathfrak{m}^{p_{i-1} - q - \sum_{\ell=j}^{i-1} v(y_\ell) + 1}$ is contained in $\mathrm{Tr}^{i-j}_{\mathbf{p}_j(q)}(I_t : (y_1 \dots y_j)|\mathbf{y}_j)$. We also have

$$\begin{aligned}
\mathrm{Tr}^{i-j}_{\mathbf{p}_j(q)}(I_t : (y_1 \dots y_j)|\mathbf{y}_j) &\subset \mathrm{Tr}^{i-j}_{(\mathbf{p}_j-1)(q)}(I_t : (y_1 \dots y_j)|\mathbf{y}_j) \\
\mathrm{Tr}^{i-j}_{\mathbf{p}_j(q)}(I_t : (y_1 \dots y_j)|\mathbf{y}_i) &\subset \mathrm{Tr}^{i}_{\mathbf{p}(q)}(I_t|\mathbf{y}) \\
\mathrm{Tr}^{i-j}_{(\mathbf{p}_j-1)(q)}(I_t : (y_1 \dots y_j)|\mathbf{y}_i) &\subset \mathrm{Tr}^{i}_{(\mathbf{p}-1)(q)}(I_t|\mathbf{y}),
\end{aligned}$$

so in particular $\mathfrak{m}^{p_{i-1} - q - V^{i-1}_j + 1}$ is contained in the four ideals involved. Using the second part of proposition 2.11 we get

$$\begin{aligned}
\mathrm{Tr}^i_{\mathbf{p}(q)}(I_t|\mathbf{y}) &= \mathrm{Tr}^i_{\mathbf{p}(q)}(I_t|\mathbf{y}) + \mathfrak{m}^{p_{i-1} - q - V^{i-1}_j + 1} = \\
&= \mathrm{Tr}^{i-j}_{\mathbf{p}_j(q)}(I_t : (y_1 \dots y_j)|\mathbf{y}_j) + \mathfrak{m}^{p_{i-1} - q - V^{i-1}_j + 1} = \\
&= \mathrm{Tr}^{i-j}_{(\mathbf{p}_j-1)(q)}(I_t : (y_1 \dots y_j)|\mathbf{y}_j) + \mathfrak{m}^{p_{i-1} - q - V^{i-1}_j + 1} = \\
&= \mathrm{Tr}^i_{(\mathbf{p}-1)(q)}(I_t|\mathbf{y}) + \mathfrak{m}^{p_{i-1} - q - V^{i-1}_j + 1} = \mathrm{Tr}^i_{(\mathbf{p}-1)(q)}(I_t|\mathbf{y}),
\end{aligned}$$

the third equality being a consequence of the second hypothesis and proposition 2.4. Now the result follows by 2.4. $\square$

2.5 Staircases and monomial ideals

We now specialize to the two-dimensional case, so let $R = k[[x, y]]$. Given a *staircase* $E \subset \mathbb{Z}^2_{\geq 0}$, i.e., a subset satisfying $E + \mathbb{Z}^2_{\geq 0} \subset E$, and a system of parameters $f, g \in R \cong k[[x, y]]$, we denote

$$I_{E,f,g} = (f^{e_1} g^{e_2})_{(e_1,e_2)\in E}\,.$$

If $E \subset \mathbb{Z}^2_{\geq 0}$ is a staircase, the *length* of its ith stair is $\ell_E(i) = \min\{e \mid (e, i) \in E\}$, and the *height* of its ith slice is $h_E(i) = \min\{e \mid (i, e) \in E\}$. We use the first difference of ℓ as well: $\hat{\ell}_E(i) = \ell_E(i) - \ell_E(i + 1)$. When there are no steps of height > 1, i.e., if $h_E(i) \leq h_E(i + 1) + 1$ for all i, we say that E is *gentle*. We also define the total length and height of E as $\ell(E) = \ell_E(0)$ and $h(E) = h_E(0)$, and the minimal length $\hat{\ell}_{\min}(E) = \min\{\hat{\ell}_E(i) \mid 0 \leq i < h(E) - 1\}$ (for technical reasons that will become apparent in forthcoming sections, the latter does not take into account the length of the top stair).

Lemma 2.13. *For every staircase E with finite complement, and every system of parameters $f, g \in R \cong k[[x, y]]$,*

1. *$I_{E,f,g}$ is $\mathfrak{m}$-primary, and has colength $\#(\mathbb{Z}^2_{\geq 0} \setminus E)$, and*
2. *$I_{E,f,g}$ depends only on finite jets of f and g, i.e., there exist integers $a = a(E)$ and $b = b(E)$ such that $f_1 - f_2 \in \mathfrak{m}^a$, $g_1 - g_2 \in \mathfrak{m}^b$ imply $I_{E,f_1,g_1} = I_{E,f_2,g_2}$.*
3. *if E is gentle then $I_{E,f,g}$ does not depend on f, i.e., $I_{E,f_1,g} = I_{E,f_2,g}$ whenever $(f_1, g) = (f_2, g) = \mathfrak{m}$. In such a case we denote $I_{E,g} = I_{E,f_1,g}$.*

Proof. Because E has finite complement, it follows that for suitable e_1, e_2, $f^{e_1} \in I_1$ and $g^{e_2} \in I_{E,f,g}$, so

$$\mathfrak{m}^{e_1+e_2} = (f, g)^{e_1+e_2} \subset I_{E,f,g}$$

and $I_{E,f,g}$ is $\mathfrak{m}$-primary. Observe that e_1, e_2 depend only on E, not on f or g. The colength follows from the well known fact that the classes modulo $I_{E,f,g}$ of the monomials $f^{e_1} g^{e_2}$ with (e_1, e_2) *not* in E form a basis of $R/I_{E,f,g}$.

For the second claim, by symmetry, it is enough to prove that $I_{E,f,g}$ depends only on a finite jet of f. We have just seen that there is a fixed integer a such that $\mathfrak{m}^a \subset I_{E,f,g}$ for every choice of f, and we want to prove that given $f_1, f_2 \in R$ with $f_1 - f_2 \in \mathfrak{m}^a$, $I_{E,f_1,g} = I_{E,f_2,g}$. Again by symmetry it will be enough to show that for every $(e_1, e_2) \in E$, $f_1^{e_1} g^{e_2} \in I_{E,f_2,g}$. This follows from

$$f_1^{e_1} g^{e_2} - f_2^{e_1} g^{e_2} = g^{e_2}(f_1^{e_1} - f_2^{e_1}) \in g^{e_2}(f_1 - f_2).$$

Finally for the third claim, and by symmetry again, we have to see that given f_1, f_2 such that $(f_1, g) = (f_2, g) = \mathfrak{m}$, for every $(e_1, e_2) \in E$ one has $f_1^{e_1} g^{e_2} \in I_{E,f_2,g}$. But, because the staircase is gentle, it follows that if $(e_1, e_2) \in E$, then for every integer $0 \leq k \leq e_1$, $(e_1 - k, e_2 + k) \in E$, and therefore $g^{e_2}(f_2, g)^{e_1} \subset I_{E,f_2,g}$. Now since both (f_1, g) and (f_2, g) are systems of parameters, it follows that $f_1 \in (f_2, g)$, and $f_1^{e_1} g^{e_2} \in g^{e_2}(f_2, g)^{e_1} \subset I_{E,f_2,g}$. $\square$

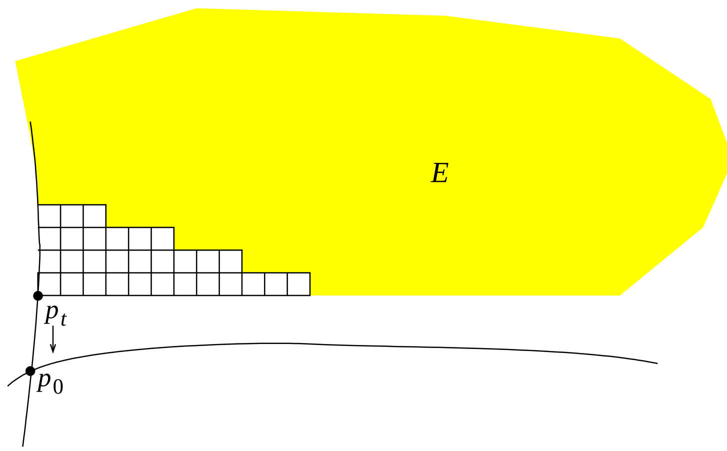

Figure 1: Example of staircase; the shaded part is E, and its complement has been drawn as a pile of boxes in staircase form. Here $E = \{(e_1, e_2) \in \mathbb{Z}^2_{\geq 0} | e_1 + 3e_2 \geq 12\}$. The corresponding ideal is $I_E = (x^3, f)^4$.

We are interested in a specific kind of families of translated monomial ideals. Fix $f = x + y + t \in R_t = k[[x, y, t]]$. For every staircase E, define

$$I_E = I_{E,x,f} = (x^{e_1} f^{e_2})_{(e_1,e_2)\in E} .$$

The sequences $\mathbf{y}$ of interest will have a fixed form as well, namely

$$\mathbf{y_m} = (y, x, \dots, x, y, x, \dots, x, y, x, \dots)$$

where $\mathbf{m} = (m_1, \dots, m_\mu) \in \mathbb{Z}^\mu_{>0}$, $y_i = y$ for $i = 1, 1+m_1, \dots, 1+m_1+\cdots+m_{\mu-1}$ and $y_i = x$ for all other $i \leq \sum m_i$. In other words, $\mathbf{y_m}$ is the concatenation of μ sequences of the form $(y, x, \dots, x)$ of lengths $m_1, \dots, m_\mu$.

The properties of staircase ideals with respect to higher order traces and residuals have been extensively studied by Évain [19], [23], in the particular case of vertical translations (roughly speaking, using $f = y + t$). His results show that traces can be computed from slices of the staircase, and residuals are obtained by deleting the same slices. This is not always the case for non-vertical translations like the ones just defined, as showed by example 2.6; the key lemma 2.12 will show that under suitable numerical conditions Évain's computations do hold in our setting as well.

For convenience, we introduce a function $\sigma_\mathbf{m}$ to count the number of x appearing in $\mathbf{y_m}$ up to the ith position, and horizontal translation of staircases.

$$\sigma_\mathbf{m}(i) = i - 1 - \max\left\{ k \,\middle|\, \sum_{j=1}^{k} m_j \leq i - 1 \right\},$$
$$\tau(E, i) = \{(e_1, e_2) \mid (e_1 + i, e_2) \in E\} .$$

The following two propositions are particular cases of the computations done by Évain in [19] and [23], and we refer the reader to these works for the proofs.

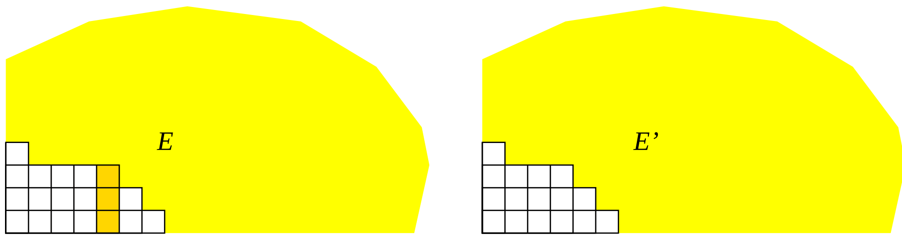

Figure 2: Example corresponding to proposition 2.15; here $p = 5$ and $i = 2$. The shaded slice in the complement to E has to be deleted to obtain E'.

Proposition 2.14. *Let $E \subset \mathbb{Z}^2_{\geq 0}$ be a staircase, $\mathbf{m} = (m_1, \ldots, m_\mu) \in \mathbb{Z}^\mu_{>0}$ a sequence of positive integers, and $I_t = I_E$, $\mathbf{y} = \mathbf{y_m}$ as defined above. Then*

1. *R_t/I_t is flat over $k[[t]]$ and over $k[[y]]$,*
2. *$I_t : y_1 \ldots y_i = I_{\tau(E, \sigma_{\mathbf{m}}(i))}$,*
3. *For every $q \geq p \geq 1$, $\mathbb{R}\mathrm{es}_p(I_t + (t^q)|x) = \mathbb{R}\mathrm{es}_p(I_t|x) = I_{\tau(E,1)} + (t^p)$, and $\mathrm{Tr}_p(I_t + (t^q)|x) = \mathrm{Tr}_p(I_t|x) = (y^{h(E)}, x)/(x)$,*

Proposition 2.15. *Let $E \subset \mathbb{Z}^2_{\geq 0}$ be a gentle staircase with finite complement. Then for every couple of integers $q > p \geq 1$,*

1. *$\mathrm{tr}_p(I_E + (t^q)|y) = \mathrm{tr}_p(I_E|y) = h_E(p-1)$,*
2. *if $p = \ell_E(i)$ for some i then $\mathrm{Res}_p(I_E + (t^q)|y) = \mathrm{Res}_p(I_t|y) = (I_{E'} + (t))/(t)$, where E' is the only staircase with*

$$\hat{\ell}_{E'}(j) = \begin{cases} \hat{\ell}_E(j) - 1 & \text{if } j = i \\ \hat{\ell}_E(j) & \text{if } j \neq i. \end{cases}$$

E' is the staircase obtained from E by deleting (from its complement) a slice of height $i + 1 = h_E(p-1) = \mathrm{tr}_p(I_E|y)$ and moving everything to the left.

Corollary 2.16. *If $E \subset \mathbb{Z}^2_{\geq 0}$ is a gentle staircase, then $\widetilde{\mathrm{tr}}_p(I_E + (t^q)|x) = \widetilde{\mathrm{tr}}_p(I_E|x) = h(E)$ and $\widetilde{\mathrm{tr}}_p(I_E + (t^q)|y) = \widetilde{\mathrm{tr}}_p(I_E|y) = h_E(p-1)$ for all $q > p \geq 1$.*

Theorem 2.17. *Let $E \subset \mathbb{Z}^2_{\geq 0}$ be a given staircase with finite complement, and let $\mathbf{m} = (m_1, \ldots, m_\mu), \mathbf{tr} = (tr_1, \ldots, tr_\mu) \in \mathbb{Z}^\mu_{>0}$ be given sequences, with $tr_1 < tr_2 < \cdots < tr_\mu$. Define $n_i = \sum_{j<i}(m_j - 1)$. Assume that*

1. *$\hat{\ell}_E(tr_i - 1) \geq tr_i + 1 \quad \forall i < \mu$,*
2. *$\ell_E(tr_i - 1) - \ell_E(tr_{i+1} - 1) \geq h_E(n_i) \quad \forall i < \mu$,*
3. *$\ell_E(tr_\mu - 1) > n_\mu$,*
4. *if $\ell_E(tr_\mu) > n_\mu$ then $\hat{\ell}_E(tr_\mu - 1) \geq tr_\mu + 1$.*

Then there exists $\mathbf{p} = (p_1, \ldots, p_m)$ with $m = \sum m_i$ such that

1. *for* $j = 1, \dots, \mu$, $\operatorname{tr}_{\mathbf{p}}^{j+n_j}(I_E|\mathbf{y_m}) = tr_j$,

2. *for* $j + n_j < i \le j + n_{j+1}$, $\operatorname{tr}_{\mathbf{p}}^{i}(I_E|\mathbf{y_m}) = h_E(i-j-1)$,

3. $\operatorname{Res}_{\mathbf{p}}^{m}(I_E|\mathbf{y}) = (I_{E'})_0$, *where* $E' = \tau(E^\flat, m-\mu)$ *and* $E^\flat$ *is the staircase with finite complement that has*

$$\hat{\ell}_{E^\flat}(tr_i - 1) = \hat{\ell}_E(tr_i - 1) - 1,\ i = 1, \dots \mu,$$
$$\hat{\ell}_{E^\flat}(j) = \hat{\ell}_E(j),\ \textit{whenever } j+1 \notin \mathbf{tr}.$$

In particular the number of conditions is preserved.

E' is the staircase obtained from E by deleting the leftmost $m-\mu$ slices, and further μ slices of heights $tr_1, tr_2, \dots, tr_\mu$.

Proof. For simplicity denote $\mathbf{y} = \mathbf{y_m}$. Define $\mathbf{p}$ as follows. $p_{n_j+j} = \ell_E(tr_j - 1) - n_j$ for $1 \le j \le \mu - 1$, $p_{n_\mu+\mu} = \ell_E(tr_\mu - 1) - n_\mu$ if $\ell_E(tr_\mu) > n_\mu$, $p_{n_\mu+\mu} = 1$ otherwise; $p_{n_j+j+1} = \ell_E(tr_j - 1) - n_j - h_E(n_j)$ for $1 \le j \le \mu - 1$ (so for instance $p_1 = \ell_E(tr_1 - 1)$ and $p_2 = \ell_E(tr_1 - 1) - h(E)$), $p_i = 1$ for $i > n_\mu + \mu$ and $p_i = p_{i-1} - 1$ for all other i. The numerical hypotheses 2 and 3 on the lengths of the stairs of E guarantee that with this definition $p_1 > \dots > p_{n_j+j} \ge p_{n_j+j+1} + h_E(n_j) \ge \dots \ge p_m$. Then we claim that 1, 2 and 3 hold.

To begin with, let us prove claims 1 and 2 for $1 \le i < n_\mu + \mu$, and for $i = n_\mu + \mu$ if $\ell_E(tr_\mu) > n_\mu$. Due to proposition 2.14, for all i and j with $j + n_j < i \le j + 1 + n_{j+1}$,

$$\widetilde{\operatorname{tr}}_{\mathbf{p}_{j+n_j}}^{i-j-n_j}(I_t : (y_1 \dots y_{j+n_j})|\mathbf{y}_{j+n_j}) = \widetilde{\operatorname{tr}}_{\mathbf{p}_{j+n_j}}^{i-j-n_j}(I_{\tau(E,\sigma_{\mathbf{m}}(j+n_j))}|\mathbf{y}_{j+n_j}) = \\ = \widetilde{\operatorname{tr}}_{\mathbf{p}_{i-1}}^{1}(I_{\tau(E,\sigma_{\mathbf{m}}(i-1))}|y_i). \quad (9)$$

If $i \le j + n_{j+1}$ then (9) can be evaluated using proposition 2.14, which gives

$$\widetilde{\operatorname{tr}}_{\mathbf{p}_{i-1}}^{1}(I_{\tau(E,\sigma_{\mathbf{m}}(i-1))}|y_i) = \operatorname{tr}_{p_i}(I_{\tau(E,i-j-1)}|y_i) = \\ = h(\tau(E, i-j-1)) = h_E(i-j-1).$$

On the other hand, if $i = j + 1 + n_{j+1}$, then using 2.15 we get

$$\widetilde{\operatorname{tr}}_{\mathbf{p}_{i-1}}^{1}(I_{\tau(E,\sigma_{\mathbf{m}}(i-1))}|y_i) = \operatorname{tr}_{p_i}(I_{\tau(E,i-j-1)}|y_i) = h_{\tau(E,n_{j+1})}(p_{j+1+n_{j+1}} - 1) = \\ = h_E(\ell_E(tr_{j+1} - 1) - n_{j+1} + n_{j+1} - 1) = tr_{j+1}.$$

In both cases the result is bounded above by $h_E(n_j)$ and therefore

$$\mathfrak{m}^{h_E(n_{j-1})} \subset \operatorname{Tr}_{\mathbf{p}_{j+n_j}}^{i-j-n_j}(I_t : (y_1 \dots y_{j+n_j})|\mathbf{y}_{j+n_j}) \subset \operatorname{Tr}_{\mathbf{p}}^{i}(I_t|\mathbf{y}),$$

But the definition of $\mathbf{p}$ gives that $p_{j+n_j} - p_i \ge h_E(n_j) + i - j - n_j - 1$, and we also have $(i - j - n_j) = \sum_{\ell=j+n_j}^{i-1} v(y_\ell)$, therefore

$$\mathfrak{m}^{p_{j+n_j} - p_i + 1 - \sum_{\ell=j+n_j}^{i-1} v(y_\ell)} \subset \mathfrak{m}^{h_E(n_{j-1})}$$

and by 2.11, $\operatorname{tr}_{\mathbf{p}}^{i}(I_t|\mathbf{y}) = \operatorname{tr}_{\mathbf{p}_{j+n_j}}^{i-j-n_j}(I_t : (y_1 \dots y_{j+n_j})|\mathbf{y}_{j+n_j})$ is as claimed.

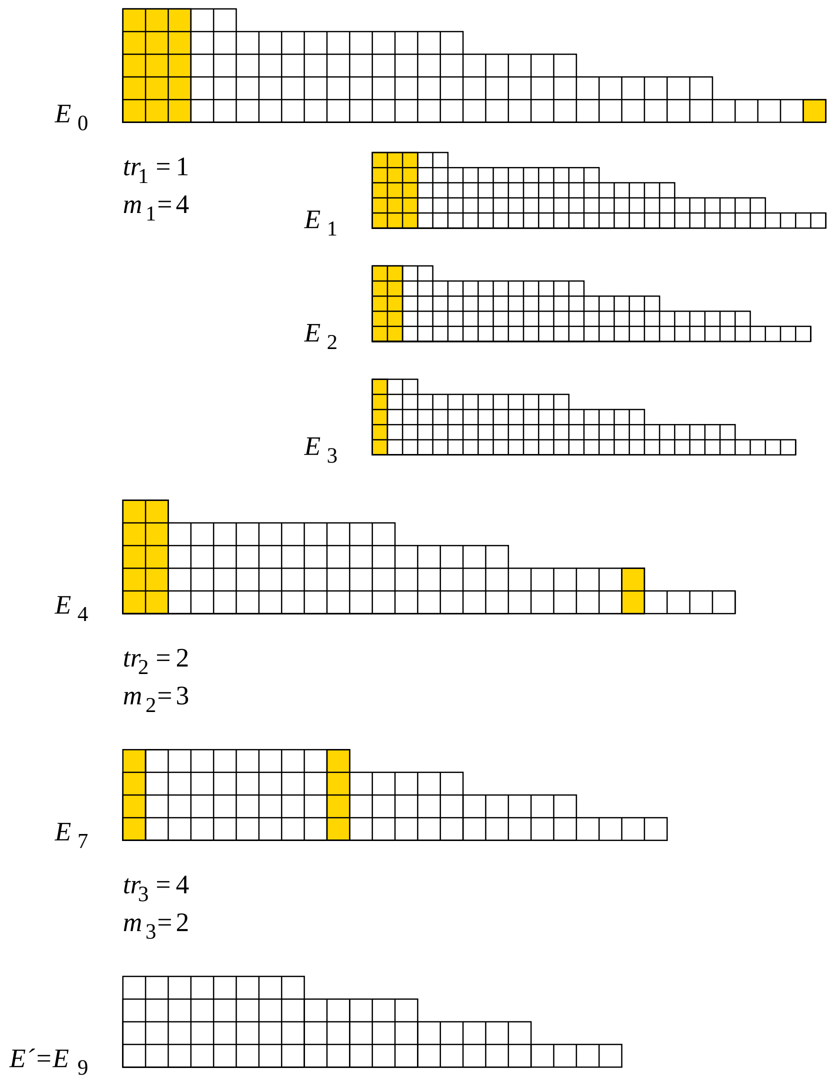

Figure 3: Example of the computation of residuals in the proof of theorem 2.17. Given $E = E_0$, $\mathbf{tr} = (1, 2, 4)$ and $\mathbf{m} = (4, 3, 2)$, the complements to the staircases E_i are as shown. For simplicity, starting with $i = 4$ we only show the steps $j + n_j$ of the sequence. Shaded, slices to erase.

Before considering the cases with $i > \mu + n_\mu$ let us compute $\mathrm{Res}^i_{\mathbf{p}}(I_E|\mathbf{y})$ for $1 \le i \le n_\mu + \mu$. We claim that $\mathrm{Res}^i_{\mathbf{p}}(I_E|\mathbf{y}) = (I_{E_i})_0$, where $E_0 = E$, E_{j+n_j} is the only staircase with finite colength and

$$\hat{\ell}_{E_{j+n_j}}(k) = \begin{cases} \hat{\ell}_{E_{j+n_j-1}}(k) - 1 & \text{if } k = tr_j - 1 \\ \hat{\ell}_{E_{j+n_j-1}}(k) & \text{if } k \ne tr_j - 1. \end{cases}$$

and $E_i = \tau(E_{i-1}, 1)$ for $j + n_j < i \le j + n_{j+1}$. Remark that with this definition, $E_m = E'$ (see figure 3). For $i = 0$ there is nothing to prove. For $i > 0$ we proceed recursively, so assume we have proved that $\mathrm{Res}^{i-1}_{\mathbf{p}}(I_E|\mathbf{y}) = (I_{E_{i-1}})_0$. Now from proposition 2.11, part 1 we deduce that $\mathrm{Res}^i_{\mathbf{p}}(I_t|\mathbf{y}) + \mathfrak{m}^{p_i-1} = (I_t : (y_1 \dots y_{i-1}))_0 + \mathfrak{m}^{p_i-1} = (I_{E_{i-1}})_0 + \mathfrak{m}^{p_i-1}$, and one has $\mathrm{Res}^i_{\mathbf{p}}(I_t|\mathbf{y}) \subset \mathrm{Res}^{i-1}_{\mathbf{p}}(I_t|\mathbf{y}) : y_i$ as well. Therefore $\mathrm{Res}^i_{\mathbf{p}}(I_t|\mathbf{y}) \subset (I_{E_{i-1}})_0 + (\mathfrak{m}^{p_i-1} \cap (I_{\tau(E_{i-1},1)})_0) = (I_{E_i})_0$, where in the case $i = j + n_j$ we use the hypothesis 1 (or hypothesis 4) as in the proof of 2.15. On the other hand, from the first two claims and the key lemma 2.12 we get that $\mathrm{res}^i_{\mathbf{p}}(I_t|\mathbf{y}) = \#(\mathbb{Z}^2_{\ge 0} \setminus E_{i-1}) - \mathrm{tr}^i_{\mathbf{p}}(I_t|\mathbf{y}) = \#(\mathbb{Z}^2_{\ge 0} \setminus E_i) = \dim_k(R/(I_{E'})_0)$, and we are done.

Now, for $i > \mu + n_\mu$ and for $i = \mu + n_\mu$ if $\ell_E(tr_\mu) \le n_\mu$, we have $p_i = 1$, hence by 2.14

$$\mathbb{R}\mathrm{es}^i_{\mathbf{p}}(I_E|\mathbf{y}) = \mathrm{Res}^i_{\mathbf{p}}(I_E|\mathbf{y})k[[t]] + (t) = I_{\tau(E_{\mu+n_\mu}, i-\mu+n_\mu)} + (t)$$

so $\mathrm{Res}^i_{\mathbf{p}}(I_E|\mathbf{y}) = (I_{E_i})_0$ in these cases as well (in particular the last claim follows) and $\mathrm{tr}^i_{\mathbf{p}}(I_E|\mathbf{y}_m) = h_E(i - \mu - 1)$ as claimed. $\square$

3 Proof of theorem 1.1

The sequence of specializations to which we apply the preceding technique in order to prove theorem 1.1 was already introduced in [45] and used in [43], [44] and [47]. It consists in introducing satellite points: one first specializes each point to be infinitely near to the previous one and then, step by step, the third point is brought to the first irreducible exceptional component (at its intersection point with the second), then the fourth, and so on. As a byproduct, the result we obtain is slightly stronger, as it shows regularity of linear systems defined by a more general class of *cluster* schemes Z. Their ideals are obtained as follows. If $\mathbb{P}^2 = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_n$ is a sequence of blowing ups centered at the points $p_1, \dots, p_n$, π denotes their composition, and E_i is the exceptional divisor in X_n above p_i for each i, then

$$\mathcal{I} = \pi_*(\mathcal{O}_{X_n}(-eE_1 - \cdots - eE_n))$$

is the ideal sheaf defining the cluster scheme that has each point p_i with multiplcity e. We assume the clusters are consistent, i.e., $\mathcal{O}_{X_n}(-eE_1 - \cdots - eE_n)$ cuts non-negatively each irreducible component of the E_i. Under these hypotheses there can be no satellites among the p_i, i.e., each point belongs to at most one irreducible exceptional component. The information of proximities satisfied by the points (i.e., which points belong to which exceptional components) can be encoded into Enriques diagrams or proximity matrices [7]. Clusters with the same diagram $\mathbf{D}$ (or with the same matrix) are parameterized by an irreducible quasiprojective variety $Cl(\mathbf{D})$ [46], and the expression "general clusters with diagram $\mathbf{D}$" refers to clusters parameterized by a Zariski open subset of $Cl(\mathbf{D})$.

Theorem 3.1. *Let n, e be positive integers, with $\sqrt{n} \geq 2e$ and $e > 2$. Then for every consistent weighted Enriques diagram $\mathbf{D}$ with exactly n vertices, all of multiplicity e, general weighted clusters of type $\mathbf{D}$ on the projective plane have maximal rank in all degreees.*

The technical hypothesis $e > 2$ is not really restrictive: for $e \leq 2$ the result is known to be true with no restriction on n (see [47]). Theorem 1.1 corresponds to the particular in which $\mathbf{D}$ consists of n distinct points.

Section 3.1 introduces the specializations that will be used to prove theorem 3.1. These consist in gradually increasing the contact of suitable schemes defined by monomial ideals with a curve of selfintersection -1, which in the application will be the exceptional divisor of blowing up a point. Using some additional blow ups and the results of section 2 we show how to bound the desired limits. Then we exploit results from [47] to prove the theorem in section 3.2.

3.1 Gentle staircases on blown-up surfaces

Let S be a smooth, projective, algebraic surface, and D a (-1)-curve on it. Given $p \in D$, let R be the completion of the local ring $\mathcal{O}_{S,p}$, and fix an isomorphism $R = \hat{\mathcal{O}}_{S,p} \cong k[[x,y]]$ such that $y = 0$ is a local equation for D at p. If no confusion is likely, the maximal ideals of $\mathcal{O}_{S,p}$ and $k[[x,y]]$ will be both denoted by $\mathfrak{m}$. By the second part of lemma 2.13, every possible monomial ideal $I_{E,f,g} \subset k[[x,y]]$can be defined by $f, g \in \mathcal{O}_{S,p}$. $I_{E,f,g} \cap \mathcal{O}_{S,p}$ is primary with respect to the maximal ideal too, of the same colength $\#(\mathbb{Z}^2_{\geq 0} \setminus E)$.

Let now $C \subset S$ a curve through p, and $g \in \mathcal{O}_{S,p}$ a local equation for C. If E is a gentle staircase, $I_{E,f,g}$ does not depend on f by 2.13; then we denote $I_{E,g} = I_{E,f,g}$ and define $\mathcal{I}_{p,E,C}$ or $\mathcal{I}_{p,E,g}$ to be the ideal sheaf with cosupport at p and stalk $I_{E,g} \cap \mathcal{O}_{S,p}$, and $Z_{p,E,C} \subset S$ or $Z_{p,E,g} \subset S$ the zeroscheme it defines.

If L is a divisor such that $\operatorname{length}(Z_{p,E,C} \cap D) > L \cdot D$ then D is a fixed part of all curves in $|L|$ that contain $Z_{p,E,C}$, if they exist. In order to compute $\operatorname{length}(Z_{p,E,C} \cap D)$ we introduce a couple of definitions. Given an integer $r \geq 0$, we denote $h^r_E = \min\{i | \hat{\ell}_E(i) \leq r\}$. If the function $\hat{\ell}_E : \mathbb{Z}_{\geq 0} \longrightarrow \mathbb{Z}_{\geq 0}$ is non-increasing in the interval $[h^r_E, \infty)$ then we say that E is r-gentle. Observe that for $r > 0$ r-gentle implies $(r-1)$-gentle, and 0-gentle implies gentle.

Lemma 3.2. *Let $\pi : S' \longrightarrow S$ be the blow-up of p, let D_p be the exceptional divisor and $\tilde{D}$ the strict transform of D. Let C be a curve going through p and smooth at p, $\tilde{C}$ its strict transform on the blow-up and $p' = \tilde{C} \cap D_p$. If E is a 1-gentle staircase with finite complement, then $\mathcal{I}_{p,E,C} = \pi_*(\mathcal{I}_{p',\tilde{E},\tilde{C}} \otimes \mathcal{O}_{S'}(-h(E)D_p))$, where $\tilde{E}$ is the staircase with finite complement that has*

$$\hat{\ell}_{\tilde{E}}(i) = \max\{\hat{\ell}_E(i) - 1, 0\}$$

for all i. Moreover, if E is r-gentle $r \geq 1$, then $\tilde{E}$ is $(r-1)$-gentle.

Proof. Both ideal sheaves $\mathcal{I}_{p,E,C}$ and $\pi_*(\mathcal{I}_{p',\tilde{E},\tilde{C}} \otimes \mathcal{O}_{S'}(-h(E)D_p))$ have cosupport at p, and their stalk there is primary with respect to the maximal ideal. Therefore, it will be enough to see that their extensions to the completion of $\mathcal{O}_{S,p}$ coincide.

Let $f = 0$ is a local equation of C, $y = 0$ a local equation of D and assume that the isomorphism $\hat{\mathcal{O}}_{S,p} \cong k[[x,y]]$ has been chosen in such a way that $(x, f) =$

$\mathfrak{m}$. Then $\hat{\mathcal{O}}_{S,p} \cong k[[x,f]]$, and $\hat{\mathcal{O}}_{S',p'} \cong R' = R[[f/x]] = k[[x, f/x]]$. Therefore $f/x = 0$ is a local equation of $\tilde{C}$, $x = 0$ a local equation for D_p and, if $\tilde{D}$ goes through p' (which means that $(D \cdot C)_p > 1$) then y/x is a local equation of $\tilde{D}$. Then, the stalk at p of $\pi_*(\mathcal{I}_{\tilde{E},\tilde{C}} \otimes \mathcal{O}_{S'}(-h(E)D_p))$ is $\mathcal{O}_{S,p} \cap (x^{h(E)} I_{\tilde{E},\tilde{C}})$, and its extension to R is

$$R \cap (x^{h(E)} I_{\tilde{E},\tilde{C}}) = k[[x,f]] \cap \left(\left(x^{h(E)+e_1} (f/x)^{e_2} \right)_{(e_1,e_2) \in \tilde{E}'} \right) =$$
$$= \left(x^{h(E)+e_1-e_2} f^{e_2} \right)_{(e_1,e_2) \in \tilde{E}'} \subset k[[x,f]] = R.$$

Now, if E is 1-gentle, it is immediate to check that $(h(E) + e_1 - e_2, e_2) \in E$ if and only if $(e_1, e_2) \in \tilde{E}'$, and we are done. The claim on $(r-1)$-gentleness of $\tilde{E}'$ is immediate from the definitions. $\square$

We can iterate this process, by blowing-up p', then $p'' = \tilde{C} \cap D_{p'}$, and so on, if the staircase is gentle enough:

Corollary 3.3. *Let E be a r-gentle staircase with finite complement, C a curve going through p and smooth at p, and $(C \cdot D)_p = s \geq r$. Let $p, p', \dots, p^r$ be the first $r+1$ points on C infinitely near to p, and S_r the surface obtained by blowing up $p, p', \dots, p^{r-1}$, on which p^r lies. Then $\mathcal{I}_{p,E,C} = \pi_*(\mathcal{I}_{p^r,\tilde{E},\tilde{C}} \otimes \mathcal{O}_{S_r}(-d_0 D_p - d_1 D_{p'} - \cdots - d_{r-1} D_{p^{r-1}}))$, where $\tilde{E}$ is the staircase with finite complement that has*

$$\hat{\ell}_{\tilde{E}}(i) = \max\{\hat{\ell}_E(i) - r, 0\}$$

for all i, $d_i = \max\{j | \hat{\ell}_E(j) > i\} + 1$ and D_{p^i} denotes the total transform in S_r of the ith exceptional divisor. If moreover $\hat{\ell}_{\min}(E) \geq r$ then $d_i = h_E(i)$.

Corollary 3.4. *Let $E \subset \mathbb{Z}^2_{\geq 0}$ be a staircase with finite complement, and C a curve going through p and smooth at p such that $(C \cdot D)_p = r$. If E is r-gentle then* $\operatorname{length}(Z_{p,E,C} \cap D) = \ell_E(h^r_E) + r h^r_E$. *If moreover $\hat{\ell}_{\min}(E) \geq r$ then* $\operatorname{length}(Z_{p,E,C} \cap D) = \sum_{i=0}^{r-1} h_E(i)$.

Proof. Let as before $p, p', \dots, p^r$ be the first $r+1$ points on C infinitely near to p, and S_r the surface obtained by blowing up $p, p', \dots, p^{r-1}$. $p^r \in S_r$ does not belong to the strict transform of D because $(C \cdot D)_p = r$, and due to Corollary 3.3, $\mathcal{I}_{p,E,C} = \pi_*(\mathcal{I}_{p^r \tilde{E},\tilde{f}} \otimes \mathcal{O}_{S_r}(-d_0 D_p - d_1 D_{p'} - \cdots - d_{r-1} D_{p^{r-1}}))$, where $d_i = \max\{j | \hat{\ell}_E(j) > i\} + 1$. Then by the projection formula, $\operatorname{length}(Z_{p,E,C} \cap D) = \sum_{i=0}^{r-1} d_i = \ell_E(h^r_E) + r h^r_E$, as wanted.

If moreover $\hat{\ell}_{\min}(E) \geq r$ then $d_i = h_E(i)$ so by the projection formula again $\operatorname{length}(Z_{p,E,C} \cap D) = \sum_{i=0}^{r-1} h_E(i)$. $\square$

Remark 3.5. It is worth noting that if $\hat{\ell}_{\min}(E) \geq r + 1$ then E is r-gentle and the previous two corollaries apply.

Given a triple (L, E, r), where L is a divisor class on S, r is a positive integer and E is a r-gentle staircase with finite complement, we say that a linear system Σ on S has type (L, E, r) if there is a curve C through p, smooth at p and with $(C \cdot D)_p = r$, such that $\Sigma = \mathbb{P}(H^0(\mathcal{I}_{p,E,C} \otimes L))$. If $L \cdot D \geq \ell_E(h^r_E) + r h^r_E$, then the type (L, E, r) is called *consistent*.

Given a family of curves C_t through p, the intersection number $(C_t \cdot D)_p$ may depend on the parameter t, i.e., one may have $(C_t \cdot D)_p = r$ and $(C_0 \cdot D)_p = r+1$, for instance. Then one obtains a family of linear systems Σ_t, $t \neq 0$ of type (L, E, r) whose limit when $t \mapsto 0$ is of different type.

Lemma 3.6. *Let (L, E, r) be a consistent type with $r > 1$. Every linear system of type (L, E, r) contains as a sublinear system the moving part of the limit of a family of linear systems of type $(L, E, r-1)$.*

Proof. Fix local coordinates (x, y) such that $y = 0$ is a local equation for D. For every curve C on S going through p with $(C \cdot D)_p = r$, let $f \in \mathcal{O}_{S,p} \subset \hat{\mathcal{O}}_{S,p} = k[[x, y]]$ be a local equation for C.

For $t \neq 0$, $f + tx^{r-1}$ is a local equation at p of a curve C_t with $(C_t \cdot D)_p = r-1$. Define $\Sigma_t = \mathbb{P}(H^0(\mathcal{I}_{p,E,C_t} \otimes L))$. Then it is clear by the definitions that

$$\lim_{t \mapsto 0} \Sigma_t \subset \Sigma := \mathbb{P}(H^0(\mathcal{I}_{p,E,C} \otimes L)). \qquad \square$$

Theorem 3.7. *Let (L, E, r) be a consistent type such that $\hat{\ell}_{\min}(E) \geq r + h_E^r + 1$ and $h_E^r \geq 2$. There exist an integer $\mu \geq 0$ and an $(r+1)$-gentle staircase E' with*

1. $\#\left(\mathbb{Z}^2_{\geq 0} \setminus E'\right) + \mu(L \cdot D) + \binom{\mu+1}{2} = \#\left(\mathbb{Z}^2_{\geq 0} \setminus E\right)$,
2. $\tau(E, \mu r) \subset E' \subset \tau(E, \mu(r+1))$,
3. $\ell(E') = \ell(E) - \mu(r+1)$,
4. $\hat{\ell}_{\min}(E') \geq \hat{\ell}_{\min}(E) - 1$,
5. *if* $\ell_E(h(E)-1) > \mu r+1$ *and* $\mu \geq 1$ *then* $h(E') = h(E)$ *and* $\ell_{E'}(h(E')-1) = \ell_E(h(E)-1) - (\mu r + 1)$,

such that $(L - \mu D, E', r+1)$ is consistent and every linear system of type $(L - \mu D, E', r+1)$ contains as a sublinear system the moving part of a limit of linear systems of type (L, E, r).

Proof. For every integer $i > 0$, consider the following quantities:

$$s_i = \sum_{j=r(i-1)}^{ir-1} h_E(j),$$
$$tr_i = L \cdot D + i - s_i,$$

It is clear that $s_1 \geq s_2 \geq \dots$, and therefore $tr_1 < tr_2 < \dots$. Let $\mu = \max\{i \mid \ell_E(tr_i - 1) > ri\}$, $\mathbf{tr} = (tr_1, tr_2, \dots, tr_\mu)$. If $\mu = 0$ then either $tr_1 > h(E)$ or $tr_1 = h(E)$ and $\ell_E(h(E)-1) < r$, in which case $h_E^r = h(E) - 1$; in both cases $(L, E, r+1)$ is consistent and the claims follow from lemma 3.6 setting $E' = E$.

So assume $\mu \geq 1$. We claim that the staircase $E' = \tau(E^\flat, \mu r)$, satisfies the stated conditions, where $E^\flat$ is the staircase with finite complement that has

$$\hat{\ell}_{E^\flat}(tr_i - 1) = \hat{\ell}_E(tr_i - 1) - 1, \; i = 1, \dots \mu,$$
$$\hat{\ell}_{E^\flat}(j) = \hat{\ell}_E(j), \; \text{whenever } j + 1 \notin \mathbf{tr}.$$

E' is obtained from E by deleting the "leftmost" μr slices, and further μ slices of heights $tr_1, tr_2, \ldots, tr_\mu$ (the hypothesis on $\hat{\ell}_{\min}$ guarantees that such slices exist and that this description is correct). Thus, claims 1, 2, 3, 4 and 5 follow.

Moreover, as $\hat{\ell}_{\min}(E') \geq \hat{\ell}_{\min}(E)-1 \geq r+h_E^r > r+1$, E' is $(r+1)$-gentle and for every C through p with $C \cdot D = r+1$, $\operatorname{length}(Z_{p,E',C} \cap D) = \sum_{i=0}^{r} h_{E'}(i)$, which by 2 is at most equal to $\sum_{i=\mu r}^{(\mu+1)r} h_E(i)$ and by the definition of μ this is at most $\mathcal{L} \cdot D + \mu$. Therefore $(L-\mu D, E', r+1)$ is consistent.

It remains to be seen that every linear system of type $(L-\mu D, E', r+1)$ contains as a sublinear system the moving part of a limit of linear systems of type (L, E, r) (the fixed part being μD). So let C be a curve on S going through p, with $(C \cdot D)_p = r+1$, and assumme that local coordinates (x,y) have been chosen in $\mathcal{O}_{S,p}$ in such a way that $y=0$ is a local equation for D, and $f \in \mathcal{O}_{S,p} \subset \hat{\mathcal{O}}_{S,p} = k[[x,y]]$ is a local equation for C. We need to prove that $\Sigma = \mathbb{P}(H^0(\mathcal{I}_{p,E',C} \otimes (L-\mu D))$ contains as a sublinear system the moving part of a limit of linear systems of type (L,E,r).

For every $t \neq 0$, $f+tx^r$ is a local equation at p of a curve C_t with $(C_t \cdot D)_p = r$. Define $\Sigma_t = \mathbb{P}(H^0(\mathcal{I}_{p,E,C_t} \otimes L))$. We claim that

$$\lim_{t \mapsto 0} \Sigma_t \subset \Sigma + \mu D. \tag{10}$$

The first r points on C_t infinitely near to p lie on D as well, so they do not depend on t; denote them $p, p', \ldots, p^{r-1}$, and let $\pi : S_r \to S$ be the blowing up of these points. The $(r+1)$th point on C_t infinitely near to p depends on t; let it be $p_t^r \in S_r$. We shall compute the limit of the Σ_t on S_r rather than on S. Indeed, corollary 3.3 shows that $\mathcal{I}_{p,E,C_t} = \pi_*(\mathcal{I}_{p_t^r,\tilde{E},\tilde{C}_t} \otimes \mathcal{O}_{S_r}(-d_0 D_p - d_1 D_{p'} - \cdots - d_{r-1} D_{p^{r-1}}))$, where $\tilde{E}$ is the staircase with finite complement that has

$$\hat{\ell}_{\tilde{E}}(i) = \max\{\hat{\ell}_E(i) - r, 0\}$$

for all i and $d_i = h_E(i)$. Setting $\Sigma'_t = \mathbb{P}(\mathcal{I}_{p_t^r,\tilde{E},\tilde{C}_t} \otimes \mathcal{O}_{S_r}(L - d_0 D_p - d_1 D_{p'} - \cdots - d_{r-1} D_{p^{r-1}}))$, it is clear that $\Sigma_t \overset{\pi_*}{\cong} \Sigma'_t$, and

$$\lim_{t \mapsto 0} \Sigma_t = \pi_* \left(\lim_{t \mapsto 0} \Sigma'_t \right). \tag{11}$$

On S_r, the point p_0^r belongs to the strict transforms $\tilde{C}$ and $\tilde{D}$ of the curves C and D respectively, and to the exceptional divisor $D_{p^{r-1}}$; at p_0^r, $\tilde{D}$, $D_{p^{r-1}}$ and $\tilde{C}$ are pairwise transverse. Thus, there exist $x_r, y_r \in \hat{\mathcal{O}}_{S_r,p_0^r}$ local parameters such that $y_r = 0$, $x_r = 0$, and $x_r + y_r = 0$ are local equations of $\tilde{D}$, $D_{p^{r-1}}$ and $\tilde{C}$ respectively. Then $f_t = x_r + y_r + t = 0$, for t in a neighbourhood of 0, is an equation of C_t in a neighbourhood of p_0^r.

Let $I_{\tilde{E}} = (x_r^{e_1} f_t^{e_2})_{(e_1,e_2) \in \tilde{E}}$, as in section 2.5. Define also for $i = 1, \ldots, \mu$,

$$\begin{aligned} l_i &= \ell_E(h_E(ri) - 1), \\ m_i &= \min\{r, ri - l_{i-1}, l_i - ri\}, \end{aligned}$$

$\mathbf{m} = (m_1, m_2, \ldots, m_\mu)$, $m = \sum m_i$, and $n_i = \sum_{j<i}(m_j - 1)$.

It is not difficult to check that the hypotheses of theorem 2.17 are satisfied for ideals defined by the staircase $\tilde{E}$, with $\mathbf{y} = \mathbf{y_m} = (y, x, \ldots, x, y, x, \ldots, x, y, x, \ldots)$

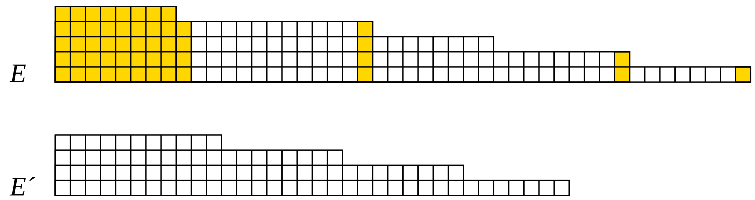

Figure 4: Example of the computation of E' in theorem 3.7. The figure shows the complement to a given staircase E; the shaded part has to be erased to obtain the complement to E' if $r = 3$ and $L \cdot D = 15$. Note that the staircases $\tilde{E}$ and $\tilde{E}'$ of the proof coincide, in this example, with the staircases E_0 and E_9 shown in figure 3.

as in section 2.5. Moreover, the staircase $\tilde{E}'$ given by 2.17 satisfies $\hat{\ell}_{\tilde{E}'}(i) = \max\{\hat{\ell}_{E'}(i)-r, 0\}$. Therefore by 3.3 one obtains $\mathcal{I}_{p,E',C} = \pi_*(\mathcal{I}_{p_0^r,\tilde{E}',\tilde{C}} \otimes \mathcal{O}_{S_r}(-d_0'D_p - d_1'D_{p'} - \cdots - d_{r-1}'D_{p^{r-1}}))$ for $d_i' = h_{E'}(i)$. In particular

$$\begin{aligned} &\mathbb{P}(H^0(\mathcal{I}_{p,E',C} \otimes \mathcal{O}_S(L-\mu D))) = \\ &= \mathbb{P}(\pi_* H^0(\mathcal{I}_{p_0^r,\tilde{E}',\tilde{C}} \otimes \mathcal{O}_{S_r}(L - \mu D - d_0'D_p - d_1'D_{p'} - \cdots - d_{r-1}'D_{p^{r-1}})). \end{aligned} \tag{12}$$

Let $V \subset \hat{\mathcal{O}}_{S_r,p_0^r}$ be the image of the natural morphism $H^0(\mathcal{O}_{S_r}(L - d_0D_p - d_1D_{p'} - \cdots - d_{r-1}D_{p^{r-1}})) \to \hat{\mathcal{O}}_{S_r,p_0^r}$. Lemma 3.8 below shows that

$$\frac{\mathrm{Res}(V|y_1\cdots y_{i-1})}{\mathrm{Res}(V|y_1\cdots y_{i-1}) \cap (y_i)} \longrightarrow \frac{R/(y_i)}{\mathrm{Tr}^i_{\mathbf{p}}(I_t|\mathbf{y})} \tag{13}$$

is injective for $i = 1, \ldots, \mu$, so theorem 2.1 applies as well, and therefore

$$\begin{aligned} \lim_{t\mapsto 0} \Sigma_t' &= \mu D + (d_0' - d_0)D_p + \ldots (d_{r-1}' - d_{r-1})D_{p^{r-1}} + \\ &\quad + \mathbb{P}(H^0(\mathcal{I}_{p_0^r,\tilde{E}',\tilde{f}}(L - \mu D - d_0'D_p - d_1'D_{p'} - \cdots - d_{r-1}'D_{p^{r-1}}))). \end{aligned} \tag{14}$$

Now it suffices to put (11), (12) and (14) together to see that (10) holds. □

Lemma 3.8. *For each i there is a divisor class F_i on S_r such that*

1. $\mathrm{Res}(V|y_1\cdots y_{i-1}) \subset \hat{\mathcal{O}}_{S_r,p_0^r}$ *is the image of the natural morphism*

$$H^0(\mathcal{O}_{S_r}(F_i)) \xrightarrow{\rho_i} \hat{\mathcal{O}}_{S_r,p_0^r},$$

2. $F_i \cdot E_i < \mathrm{tr}^i_{\mathbf{p}}(I_t|\mathbf{y})$, *where E_i is the irreducible divisor defined locally by $y_i = 0$, that is, $E_i = \tilde{D}$ if $i = n_j + j$ for some j, and $E_i = D_{p^{r-1}}$ otherwise.*

Proof. Let us define the F_i by recurrence on i. To begin with, set $F_1 = L - d_0D_p - d_1D_{p'} - \cdots - d_{r-1}D_{p^{r-1}}$. By definition and assuming F_{i-1} satisfies the claims, it is clear that $\mathrm{Res}(V|y_1\cdots y_{i-1})$ is the image of the natural morphism

$$H^0(\mathcal{O}_{S_r}(F_{i-1} - E_{i-1})) \longrightarrow \hat{\mathcal{O}}_{S_r,p_0^r}.$$

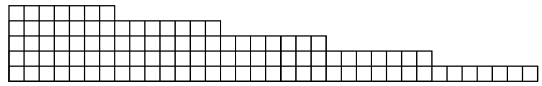

Figure 5: The complement to staircase $E_1(8,5)$.

However, the divisor class $F_{i-1} - E_{i-1}$ need not be consistent, i.e., it may intersect negatively some irreducible components of the divisors D_{p^j}, which in that case become fixed parts of $|F_{i-1} - E_{i-1}|$. We define F_i to be the consistent system obtained from $F_{i-1} - E_{i-1}$ by unloading (i.e., subtracting the fixed divisors). Clearly $H^0(\mathcal{O}_{S_r}(F_i)) \cong H^0(\mathcal{O}_{S_r}(F_{i-1} - E_{i-1}))$, the isomorphism being given by the subtraction of the fixed divisors, which do not pass through p_0^r, and therefore $\mathrm{Res}(V|y_1 \cdots y_{i-1}) \subset \hat{\mathcal{O}}_{S_r,p_0^r}$ is the image of ρ_i as stated.

Now compute F_i and $F_i \cdot E_i$. Let j be such that $n_j + j + 1 \leq i \leq n_{j+1} + j + 1$, and define

$$k_0 = \begin{cases} 0 & \text{if } h_E(r(j-1)) = h_E(rj), \\ \ell_E(h_E(r(j-1)) - 1) - r(j-1) & \text{if } h_E(r(j-1)) > h_E(rj), \end{cases}$$

and for $k = 0, \dots, r-1$,

$$d_k^i = \begin{cases} h_E(jr) & \text{if } k - k_0 \leq i - (n_j + j + 1), \\ h_E(jr) - 1 & \text{if } k - k_0 > i - (n_j + j + 1). \end{cases}$$

Remark that the definitions of d_k^i and m_i imply that, if $i = n_j + j$ then $d_k^i = h_E(r(j-1) + k)$, and if $i = n_j + j + 1$ then $d_k^i = h_E(r(j-1) + k) - 1$. Now it is an elementary unloading exercise to show that

$$F_i = L - jD - d_0^i D_p - \cdots - d_{r-1}^i D_{p^{r-1}}. \tag{15}$$

Finally, observe that by (15), if $i = n_j + j$ then $F_i \cdot E_i = F_i \cdot \tilde{D} = (L \cdot D) + j$ and for all other values of i, $F_i \cdot E_i = F_i \cdot D_{p^{r-1}} = h_E(jr) - 1$, whereas if $i = n_j + j$ then $\mathrm{tr}_{\mathbf{p}}^i(I_t|\mathbf{y}) = (L \cdot D) + j + 1$ and for all other values of i, $\mathrm{tr}_{\mathbf{p}}^i(I_t|\mathbf{y}) = h_E(i-j) > h_E(jr) - 1$, so $F_i \cdot E_i < \mathrm{tr}_{\mathbf{p}}^i(I_t|\mathbf{y})$. $\square$

3.2 Equimultiple clusters with many points

In this section, $\pi : S \to \mathbb{P}^2$ is the blowing up of $\mathbb{P}^2$ at a point, D is the exceptional divisor and (n,e) is a couple of integers with $2 < e \leq \sqrt{n}/2$.

Define $E_1(n,e) := \{(e_1, e_2) \in \mathbb{Z}^2_{\geq 0} | ee_1 + (n-1)e_2 \geq e(n-1)\}$. In other words, $E_1(n,e)$ is the staircase of height e and $\hat{\ell}_{E_1(n,e)}(i) = n-1$ for $0 \leq i < e$. Let L be a divisor class with $L \cdot D = e$, such as $\pi^*(dH) - eD$, where H is the class of a line in the plane and d a positive integer. Then $(L, E_1(n,e), 1)$ is a consistent type satisfying the requirement of theorem 3.7; let μ_1 and $E_2(n,e)$ be the integer and staircase given by 3.7. $(L - \mu_1 D, E_2(n,e), 2)$ is again a consistent type satisfying the requirement of theorem 3.7; let μ_2 and $E_3(n,e)$ be the corresponding integer and staircase. As long as the hypotheses of the theorem are satisfied, we keep using it to define integers $\mu_3, \mu_4, \dots$ and staircases $E_4(n,e), E_5(n,e), \dots$ When no confusion may arise, we denote the staircases simply $E_1, E_2, \dots$ and we also

denote $L_2 = L - \mu_1 D$, $L_3 = L_2 - \mu_2 D$, ... Let $r_{\max}(n,e)$ be the last r such that E_r is defined, i.e., $(L_{r_{\max}(n,e)}, E_{r_{\max}(n,e)}(n,e), r_{\max}(n,e))$ is a consistent type and either $h^{r_{\max}(n,e)}_{E_{r_{\max}(n,e)}(n,e)} \le 1$ or $\hat{\ell}_{\min}(E_{r_{\max}(n,e)}) \le r_{\max}(n,e) + h^{r_{\max}(n,e)}_{E_{r_{\max}(n,e)}(n,e)}$.

Lemma 3.9. *Let r be a positive integer, and denote $M = \sum \mu_i$, with the summation running over all $i \le \min\{r-1, r_{\max}(n,e)\}$.*

1. *if $r \le r_{\max}(n,e)$, then*
   (a) *$\hat{\ell}_{\min}(E_r) \ge n - r$,*
   (b) *$(r-1)(r+2)e \ge 2(n-1)(e - h(E_r))$, and*
   (c) *$\#(E_r \setminus E_1) = eM + \binom{M+1}{2}$,*
2. *If $\binom{r}{2}e + r < n-1$, then $r \le r_{\max}(n,e)$, and $\mu_1 = \mu_2 = \cdots = \mu_{r-1} = e$,*
3. *if $\binom{r}{2}e + r \ge n-1$ and $r \le r_{\max}(n,e)$, then $\#(E_r \setminus E_1) \ge e(n-1)/2$.*

Proof. Observe first that due to 3.7, claim 4, for every $1 < r \le r_{\max}(n,e)$ one has $\hat{\ell}_{\min}(E_r) \ge \hat{\ell}_{\min}(E_{r-1}) - 1$, and therefore $\hat{\ell}_{\min}(E_r) \ge n - r$. Now because of 3.7, claim 2, $E_r \supset \tau(E_1, (e - h(E_r))(n-1))$. We also have $E_r \subset \tau(E_1, \sum_{i=1}^{r-1} \mu_i(i+1))$ and $\mu_i \le e$, hence

$$\frac{(r-1)(r+2)}{2}e \ge (n-1)(e - h(E_r)).$$

Thus, claim 1 follows from because of 3.7, claim 1:

$$\#(E_r \setminus E_1) = \binom{e+M+1}{2} - \binom{e+1}{2} = eM + \binom{M+1}{2}.$$

Now because claim 1a holds, $r+1 \le r_{\max}(n,e)$ whenever $2r \le n - e - 1$, and in particular $r \le r_{\max}(n,e)$ whenever $\binom{r}{2}e + r < n-1$; in such a case moreover, due to 3.7, claim 5, $\mu_1 = \mu_2 = \cdots = \mu_{r-1} = e$.

Finally for the third claim, let r_0 be the maximal integer with $\binom{r_0}{2}e + r_0 < n-1$. The hypothesis says $r \ge r_0$, and then $M = \mu_1 + \mu_2 + \cdots + \mu_{r-1} \ge e(r_0 - 1)$. Then by claim 1c,

$$\#(E_r \setminus E_1) = eM + \binom{M+1}{2},$$

which is not less than $e(n-1)/2$ by the definition of r_0 and the inequality $M \ge (r_0 - 1)e$. □

Lemma 3.10. *For every $r \le \min\{r_{\max}(n,e), (n-1)/2\}$ such that $\binom{r}{2}e + r \ge n-1$, if $h(E_r) \ge 3$ then*

$$4r^2 + 2r + e(2en - 3e - n) \le \sqrt{3}e(n-1)(2e-1).$$

Moreover, there exists an integer $r \le r_{\max}(n,e)$ such that $h(E_r) \le 2$.

Proof. As $\binom{r}{2}e + r \ge n-1$, it follows from lemma 3.9, claim 3 that $\#(E_r \setminus E_1) \ge e(n-1)/2$. On the other hand, we also have

$$\#(E_r \setminus E_1) = \sum_i i\left(\hat{\ell}_{E_1}(i-1) - \hat{\ell}_{E_r}(i-1)\right),$$

$$\ell(E_1) - \ell(E_r) = \sum_i \left(\hat{\ell}_{E_1}(i-1) - \hat{\ell}_{E_r}(i-1)\right),$$

and by 3.7, claim 3, $\Delta_\ell := \ell(E_1) - \ell(E_r) = 2\mu_1 + 3\mu_2 + \cdots + r\mu_{r-1}$. Write $\Delta_\ell = \kappa(n-1) + \rho$, with $0 \le \rho < n-1$. As $\hat{\ell}_{E_1}(i-1) - \hat{\ell}_{E_r}(i-1) \le n-1$ for all i, an upper bound for $\#(E_r \setminus E_1)$ is

$$\#(E_r \setminus E_1) \le \sum_{i=e-\kappa+1}^{e} i(n-1) + (e-\kappa)\rho \le e\Delta_\ell - \frac{\Delta_\ell}{2}\left(\frac{\Delta_\ell}{n-1} - 1\right). \tag{16}$$

In particular $e\Delta_\ell \ge \#(E_r \setminus E_1) \ge e(n-1)/2$. On the other hand, the function $(x/2)(x/(n-1)-1)$ is increasing for $x \ge (n-1)/2$, so

$$\frac{\Delta_\ell}{2}\left(\frac{\Delta_\ell}{n-1} - 1\right) \ge \frac{\#(E_r \setminus E_1)/e}{2}\left(\frac{\#(E_r \setminus E_1)/e}{n-1} - 1\right)$$

which combined with (16) gives, denoting as before $M = \mu_1 + \mu_2 + \cdots + \mu_{r_1-1}$,

$$\Delta_\ell \ge \left(M + \frac{\binom{M+1}{2}}{e}\right)\left(1 - \frac{1}{2e} + \frac{eM + \binom{M+1}{2}}{2e^2(n-1)}\right). \tag{17}$$

On the other hand, claim 2 of theorem 3.7 tells us that

$$\begin{gathered} E_r \supset \tau(E_{r-1}, (r-1)\mu_{r-1}) \supset \\ \supset \tau(\tau(E_{r-2}, (r-2)\mu_{r-2}), (r-1)\mu_{r-1}) = \tau(E_{r-2}, (r-2)\mu_{r-2} + (r-1)\mu_{r-1}) \supset \\ \supset \cdots \supset \tau(E_1, \sum i\mu_i) = \tau(E_1, \Delta_\ell - M). \end{gathered}$$

Therefore

$$h_{E_r} \le h_{\tau(E_1,\Delta_\ell - M)} = h_{E_1}(\Delta_\ell - M) \le e - \frac{\Delta_\ell - M - (n-2)}{n-1},$$

which solving for Δ_ℓ and combining with (17), gives the bound

$$h_{E_r} \le e - \frac{eM + \binom{M+1}{2}}{e(n-1)}\left(1 - \frac{1}{2e} + \frac{eM + \binom{M+1}{2}}{2e^2(n-1)}\right) + \frac{M+n-2}{n-1}, \tag{18}$$

where the expression on the right is a decreasing function of M. On the other hand, because $(L - MD, E_r, r)$ is consistent and $\ell_{E_r}(h(E_r) - 1) \ge r$, it follows that $e + M > r(h(E_r) - 1)$; plugging the resulting bound on M into (18) and simplifying we get the following inequality:

$$\begin{gathered} \left(r^2(h(E_r)-1)^2 + r(h(E_r)-1) + e(2en - 3e - n)\right)^2 \le \\ \le e^2(n-1)^2\left(12e^2 + 4e + 1 - 8e\frac{e + r + 1 + (n-1-r)h(E_r)}{n-1}\right). \end{gathered}$$

Now, if for some $r \le (n-1)/2$, $h(E_r) \ge 3$ the previous inequality gives

$$4r^2 + 2r + e(2en - 3e - n) \le \sqrt{3}e(n-1)(2e-1). \tag{19}$$

Finally, recall from the proof of lemma 3.9 that because $\hat{\ell}_{\min}(E_r) \ge n - r$, $r_0 = \lfloor (n-e+1)/2 \rfloor \le r_{\max}(n,e)$. We shall prove that $h(E_{r_0}) \le 2$, by contradiction: suppose $h(E_{r_0}) \ge 3$. Since $\binom{r_0}{2}e + r_0 \ge n-1$ and $2r_0 \le n-1$, r_0 must satisfy the inequality (19). Then using $r_0 \ge (n-e)/2$ and the hypothesis $n \ge 4e^2$, we end up with

$$e\,(2\,e-1)\left(4\,(3-\sqrt{3})\,e^2 + 1 + \sqrt{3}\right) \le 0,$$

which is absurd for positive e. $\square$

The following result rephrases Theorem 4.3 in [47], and its corollary below follows by iteration, in an analogous way to corollary 4.6 in [47].

Theorem 3.11. *Let E be a staircase of height two and s a positive integer satisfying $\hat{\ell}_E(0) \geq s+2$ and $\ell_E(1) \geq s$, and let L be a divisor class with $L \cdot D = 2s$. Define E_1 to be the unique staircase of height (at most) two with $\ell(E_1) = \ell(E) - s - 1$, $\ell_{E_1}(1) = \ell_E(1) - s$. If $\ell_E(1) \geq 2s-1$, define furthermore E_2 to be the unique staircase of height (at most) two with $\ell(E_2) = \ell(E) - 2s - 2$, $\ell_{E_1}(1) = \ell_E(1) - 2s - 1$.*

(L, E, s) is a consistent type. If $\ell_E(1) \leq 2s-2$ then $(L-D, E_1, s+1)$ is a consistent type, and every linear system of type $(L-D, E_1, s+1)$ contains as a sublinear system the moving part of a limit of linear systems of type (L, E, s). If $\ell_E(1) \geq 2s-1$ then $(L-2D, E_2, s+1)$ is a consistent type, every linear system of type $(L-2D, E_2, s+1)$ contains as a sublinear system the moving part of a limit of linear systems of type (L, E, s).

Corollary 3.12. *Let E be a staircase of height two and s, k positive integers satisfying $\hat{\ell}_E(0) \geq s+2k-2$, $\ell(E) \geq (2s+k)(k-1)$ and $\ell_E(1) \geq (2s+k-1)(k-1)$, and let L be a divisor class with $L \cdot D = 2s$. Define $E_1 = E$ and for $k > 1$ let E_k be the unique staircase of height (at most) two with $\ell(E_k) = \ell(E) - (2s+k)(k-1)$, $\ell_{E_k}(1) = \ell_E(1) - (2s+k-1)(k-1)$.*

$(L - 2(k-1)D, E_k, s+k-1)$ is a consistent type, and if there is a linear system of this type with the expected dimension then there is a linear system of type (L, E, s) with the expected dimension.

The regular linear systems to which everything else is specialized are those of the following lemma, equivalent to lemma 4.4 of [44].

Lemma 3.13. *Let E be a staircase of height two and c a positive integer satisfying $\ell(E) > c$ and $2\ell_E(1) \leq c$. Then for every divisor class L with $L \cdot D = c$ and every integer r such that the type (L, E, r) is consistent, general linear systems of type (L, E, r) are regular.*

Lemma 3.14. *Let E be a staircase with $h(E) = 2$. Let c be a positive integer such that*

$$\hat{\ell}_E(0) + \frac{c}{2} > 1 + 3\sqrt{\ell_E(1) + \left(\frac{c}{2}\right)^2}.$$

Then for every divisor class L with $L \cdot D = c$ and every integer r such that the type (L, E, r) is consistent, general linear systems of type (L, E, r) are regular.

Proof. Note that $\ell(E) \geq \hat{\ell}_E(0) \geq 1 - c/2 + 3c/2 > c$, so if $2\ell_E(1) \leq c$ then the result follows from lemma 3.13; we assume from now on that (A) $2\ell_E(1) \geq c+1$. Also, if general linear systems of type $(L, E, r+1)$ are consistent and regular then general linear systems of type (L, E, r) are consistent and by semicontinuity regular as well.

For technical reasons we suppose that $c = 2s$ is even; let us see that this is not restrictive. If $c = 2s+1$ is odd then (A) gives $\ell_E(1) \geq s+1$ and the hypothesis of the lemma implies $\hat{\ell}_E(0) \geq 2s+1$, so for $r \leq s+1$, $h_E^r = 2$. Thus corollary 3.4 gives length$(Z_{p,E,C} \cap D) = 2r$ for all C such that $(C \cdot D)_p = r$. In particular, (L, E, r) is consistent if and only if $r \leq s$. Now we may apply theorem 2.17 or just specialize to $Z_{p,E,C}$ with $(C \cdot D)_p = s+1$ and we obtain that

it is enough to prove that general linear systems of type $(L-D, \tau(E, s+1), s+1)$ are regular. But $E' = \tau(E, s+1)$ and $c' = c+1$ satisfy

$$\hat{\ell}_{E'}(0) + \frac{c'}{2} > 1 + 3\sqrt{\ell_{E'}(1) + \left(\frac{c'}{2}\right)^2}$$

and $(L-D) \cdot D = c'$ so we have reduced to a case with even c'.

So we can assume (B) $c = 2s$, and it is not hard to see that also in this case (L, E, r) is consistent if and only if $r \leq s$. Let k be the integer such that

$$(k+s)^2 \overset{\text{(C)}}{>} \ell_E(1) + s^2 \overset{\text{(D)}}{\geq} (k+s-1)^2$$

(in particular $k \geq 1$). The hypothesis of the lemma gives (E) $\hat{\ell}_E(0) \geq 1 + 3(k+s-1) - s = 2s + 3k - 2$.

(D) can be rewritten as (D′) $\ell_E(1) \geq (2s+k-1)(k-1)$, which added to (E) gives (F) $\ell(E) \geq (2s+k+1)k-1$. Corollary 3.12 has weaker hypotheses than the inequalities (E), (F) and (D′), so it applies to the present situation. Thus we are reduced to proving that general linear systems of type $(L-2(k-1)D, E_k, s+k-1)$ are regular. Note that (E) and $s+k \geq 2$ imply (G) $\hat{\ell}_{E_k}(0) \geq s+k+1$.

We distinguish two cases. If $\ell_E(1) \leq (2s+k-1)k-s$ then $\ell_{E_k}(1) \leq s+k-1$, and (F) gives $\ell(E_k) \geq 2s+2k-1$. So lemma 3.13 proves the needed regularity.

Alternatively, if $\ell_E(1) \geq (2s+k-1)k-s+1$ then (H) $\ell_{E_k}(1) \geq s+k$. Now theorem 3.11 applies (with $s' = s+k-1$) due to (G) and (H), so it is enough to prove that general linear systems of type $(L-(2k-1)D, E'_k, s+k)$ are regular, where E'_k has height two and $\ell(E'_k) = \ell(E_k)-(s+k)$, $\ell_{E'_k}(1) = \ell_{E_k}(1)-(s+k-1)$. Adding (G) and (H) gives $\ell(E'_k) \geq 2k+2s+1$ and (C) implies $\ell_{E'_k}(1) \leq k+s-1$, so lemma 3.13 proves the needed regularity again. □

We are now ready to prove our main theorem.

Proof of 3.1. It is well known, and follows from results of [45], that by a semicontinuity argument one can restrict to the case that $\mathbf{D}$ is a unibranched diagram of exactly n free vertices of multiplicity e. Moreover, this case is equivalent to proving that, taking $\pi : S \to \mathbb{P}^2$ to be the blow-up of the plane at a point, D the exceptional divisor, and $L = \pi^*H - eD$, where H is the class of a line, general linear systems on S of type $(L, E_1(n,e), 1)$ are regular.

By theorem 3.7, every linear system of type $(L_2, E_2(n,e), 2)$ contains as a sublinear system the moving part of a limit of linear systems of type $(L, E_1(n,e), 1)$ and their expected dimensions agree by claim 1 of theorem 3.7 and [16, 2.14], so it will be enough to show that general linear systems of type $(L_2, E_2(n,e), 2)$ are regular. Iterating the process, it is enough to prove that general linear systems of type $(L_r, E_r(n,e), r)$ are regular, for some $r \leq r_{\max}(n,e)$.

Let r be the minimal integer such that $h(E_r) \leq 2$. Such an r exists by lemma 3.10. Applying 3.14, it will be enough to show that

$$\hat{\ell}_{E_r}(0) + \frac{L_r \cdot D}{2} > 1 + 3\sqrt{\ell_{E_r}(1) + \left(\frac{L_r \cdot D}{2}\right)^2}. \tag{20}$$

Now, due to theorem 3.7, claim 1,

$$\ell_{E_r}(0) + \ell_{E_r}(1) + \binom{L_r \cdot D + 1}{2} = n\binom{e+1}{2},$$

so (20) is equivalent to

$$\hat{\ell}_{E_r}(0) + \frac{L_r \cdot D}{2} > 1 + 3\sqrt{\frac{n}{2}\binom{e+1}{2} - \frac{\hat{\ell}_{E_r}(0)}{2} - \frac{L_r \cdot D}{4}},$$

and moreover, since $\ell_{E_r}(0) + \ell_{E_r}(1) \leq 3(n-1)$ it follows that $\binom{L_r \cdot D+1}{2} \geq n\binom{e+1}{2} - 3(n-1)$ which, taking into account that $e > 2$, implies $L_r \cdot D \geq e\sqrt{n} - 1/2$, so it will be enough to prove

$$\hat{\ell}_{E_r}(0) + \frac{2e\sqrt{n}-1}{4} > 1 + 3\sqrt{\frac{n}{2}\binom{e+1}{2} - \frac{\hat{\ell}_{E_r}(0)}{2} - \frac{2e\sqrt{n}-1}{8}}. \tag{21}$$

But by claim 4 of theorem 3.7, $\hat{\ell}_{E_r}(0) \geq n - r$ and the minimality of r together with lemma 3.10 give

$$4(r-1)^2 + 2(r-1) + e(2en - 3e - n) \leq \sqrt{3}e(n-1)(2e-1). \tag{22}$$

It is now a simple calculus exercise to check that if e, n, r and $\hat{\ell}_{E_r}(0)$ are integers satisfying $e > 2$, $n \geq 4e^2$, $\hat{\ell}_{E_r}(0) \geq n - r$ and (22), then (21) holds. □

Finally we prove Évain's result for a square number of points.

proof of 1.3. It is known after [21] and [6] that the result is true for n a power of four or nine, and that if it is true for n_1 and n_2, then it is true for $n_1 n_2$. So it is not restrictive to assume that s is odd and $s \geq 5$. On the other hand, due to theorem 1.1, we may assume that $e > s/2$.

Let $p_1, \ldots, p_{s^2} \in \mathbb{P}^2$ be points in general position, let Z be the union of these points taken with multiplicity e, and $\mathcal{I}_Z$ the defining ideal sheaf. By [31, 5.3] it is known that if $a \geq se + (s-3)/2$, then $H^1(\mathbb{P}^2, \mathcal{I}_Z(d)) = 0$ (the map $\rho_{n,e}(d)$ is surjective), and by 2.10 we know that if $a \leq se + (s-5)/2$, then $H^0(\mathbb{P}^2, \mathcal{I}_Z(d)) = 0$ (the map $\rho_{n,e}(d)$ is injective) so we are done. □

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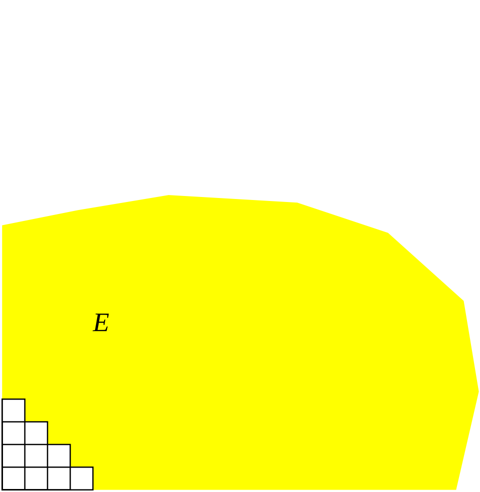
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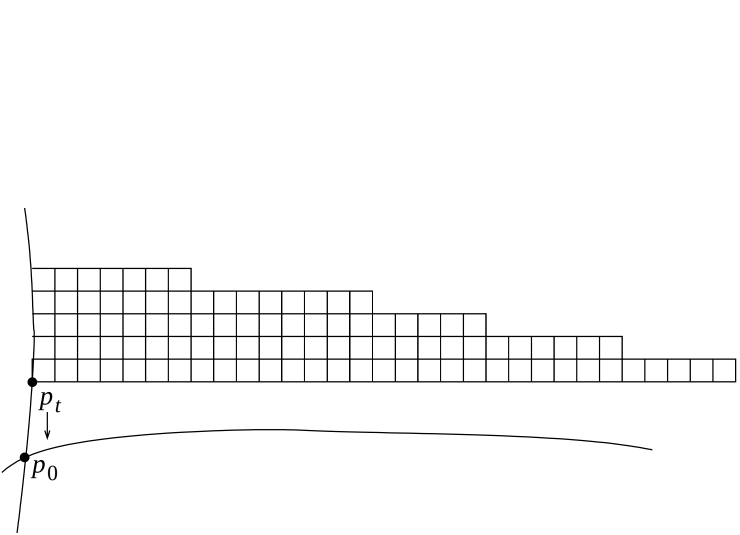

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